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ENGINEERING MECHANICS

ENGINEERING MECHANICS

DEALING MAINLY WITH KINEMATICS AND DYNAMICS
AND BEING A COMPANION VOLUME TO
D. A. LOW'S "APPLIED MECHANICS"

A TEXT-BOOK FOR ENGINEERING STUDENTS

BY

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PREFACE

THE manuscript of this book was commenced in the autumn of 1931 at the suggestion of Professor D. A. Low, the author's father, but owing to various interruptions, including the war, it has not been possible to produce the finished result until 1942.

Some explanation regarding the title and the contents may be necessary. The main title *Engineering Mechanics* has been chosen in order to avoid confusion with D. A. Low's *Applied Mechanics*, to which this book is intended to be a companion volume. The book is complete in itself, however, although reference is made to the *Applied Mechanics* in a few places to avoid unnecessary duplication. Together, the two books cover a wide field.

The book is chiefly concerned with kinematics and dynamics, including instantaneous centres, velocity and acceleration diagrams, analysis of cams, motion of rigid bodies in two dimensions, and vibrations of various kinds. In view of the importance of dimensions and dynamical similarity, a chapter is devoted to these subjects. The deflection of beams is dealt with, some knowledge of this being required in the study of certain vibration problems.

The chapter on the deflection of beams is complete in itself, but to some extent it supplements the corresponding chapter in the *Applied Mechanics*; each problem is examined from first principles and the Macaulay method is explained with examples.

The *Applied Mechanics* is still as much in demand as when it was first published, but it is believed that there are some readers who would like to have more information on the part dealing with kinematics and dynamics, and it is for this reason that the present writer has produced what he hopes will be regarded as a suitable companion volume.

There are numerous worked examples in the text, and in many instances the exercises which follow most of the chapters will be found to amplify the text. The author strongly recommends the student to work through the exercises, doing as many as time allows, since this is the only way to learn the subject.

The whole of the text, the worked examples, and the answers to the exercises have been checked by the author's son, E. D. Low, and it is hoped that the book is free from errors.

Some of the examples and exercises have been selected from examination papers set by the Universities of Cambridge and London and by the Board of Education; these are designated [C.U.], [U.L.], and [B.E.]. The author acknowledges with thanks the permission—granted by the Syndics of the Cambridge University Press, the Senate of the University of London, and the Controller of H.M. Stationery Office—to use these questions.

B. B. LOW.

February 1942.

CONTENTS

ENGINEERING MECHANICS

CHAPTER I

VELOCITY—ACCELERATION—VECTORS

1. Greek Alphabet.—A few of the Greek letters are used in this volume, and the alphabet is given here for reference.

Α	α	alpha	Ι	ι	iōta	Ρ	ρ	rho
Β	β	bēta	Κ	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	Τ	τ	tau
Δ	δ	delta	Μ	μ	mu	Υ	υ	upsīlon
Ε	ϵ	epsīlon	Ν	ν	nu	Φ	ϕ	phi
Ζ	ζ	zeta	Ξ	ξ	xi	Χ	χ	chi
Η	η	ēta	Ο	o	omīcron	Ψ	ψ	psi
Θ	θ	thēta	Π	π	pi	Ω	ω	ōmega

2. Velocity and Speed.—All motion is relative motion, and when a body is said to be fixed or at rest it is only at rest relative to another body. In engineering work, motion is generally measured relative to the frame of a machine or to the earth. The rate of change of position of a point is called the *velocity* of the point and it involves direction and sense as well as rate. The rate without consideration of direction is called *speed*. Direction is defined by saying the motion is along a certain straight line, say AB, then the sense of the direction is either from A towards B or from B towards A. If the motion of a point is along a curve, then at any instant the direction of motion is along the tangent to the curve at the point.

Speed is *uniform* or *variable* according as equal or unequal distances are traversed in equal intervals of time, however short these intervals may be. A point may have uniform speed along any path, either straight or curved, but if the

velocity is to be uniform the motion must be in a straight line since velocity involves direction as well as rate. The velocity of a point is variable when its speed is changing or when its direction of motion is changing or when both speed and direction are changing.

A moving point has *linear velocity* and the magnitude of this velocity is measured in units of length per unit time—for instance, feet per second or miles per hour. Suppose the speed, or magnitude of the velocity, is constant and is denoted by v; then if a distance s is travelled in time t,

$$s=vt \quad \text{or} \quad v=s/t.$$

If s is in feet and t is in seconds, then v is in feet per second.

A line turning about a point has *angular velocity*. If the angular velocity is uniform and its magnitude is denoted by ω, and if θ is the angle turned through in time t, then

$$\theta=\omega t \quad \text{or} \quad \omega=\theta/t.$$

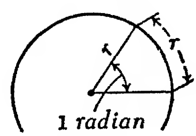

FIG. 1.

If θ is in radians and t is in seconds, then ω is in radians per second. It may be remarked here that a radian is the angle subtended at the centre of a circle by an arc whose length is equal to the radius (Fig. 1) and that $360° = 2\pi$ radians.

3. Distance-Time Graphs.—Let the distance s travelled by a point be plotted against time t, using axes OS and

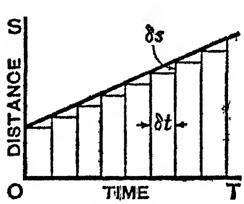

FIG. 2.

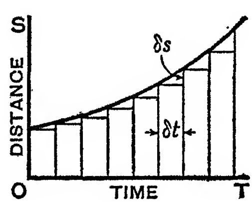

FIG. 3.

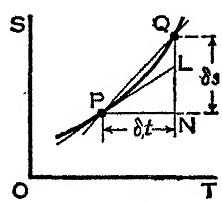

FIG. 4.

OT (Fig. 2), then if equal distances δs are traversed in equal times δt the speed is $\dfrac{\delta s}{\delta t}$, which is constant, and the

graph is a straight line. If the distances δs are not equal in all the equal intervals δt, the graph is a curve (Fig. 3) and the speed is variable since $\frac{\delta s}{\delta t}$ is different for each interval of time. For any particular interval of time the value of $\frac{\delta s}{\delta t}$ is called the *average speed* during that interval.

To find the instantaneous value of the speed at any moment, consider points P and Q near together on the curve, a short length of which is shown enlarged in Fig. 4. Draw PN and QN parallel to OT and OS, respectively, and intersecting at N. The average speed between P and Q is $\frac{QN}{PN}$ or $\frac{\delta s}{\delta t}$, where $\delta s = QN$ and $\delta t = PN$, and the nearer Q is to P the more nearly does $\frac{\delta s}{\delta t}$ represent the speed at P. Now let Q approach P, then ultimately the chord PQ becomes the tangent PL to the curve at the point P. The value to which $\frac{\delta s}{\delta t}$ approaches, as δt approaches zero, is written $\frac{ds}{dt}$ and this is the slope of the tangent and the speed at P. Therefore the speed at any point P may be found graphically by drawing a tangent to the curve and finding the slope $\frac{LN}{PN}$, measuring LN with the distance scale and measuring PN with the time scale. Although it is often difficult to draw a tangent with great accuracy, the resultant error due to inaccuracy in measurement should be made as small as possible by drawing LN at a greater distance from P than is shown in the Fig.

Sometimes a short length PQ of the curve is so nearly straight that $\frac{\delta s}{\delta t}$ gives a close approximation to the speed at the point P. When the curvature is more noticeable it will be more nearly correct to take $\frac{\delta s}{\delta t}$ as the speed at the

middle of the interval. When the speed is constant, then the graph is a straight line and at any point the value of $\frac{ds}{dt}$ is the same as the ratio $\frac{\delta s}{\delta t}$.

If the equation of the curve is known, the value of $\frac{ds}{dt}$ should be found by differentiation. This method is discussed in Chap. III.

Although the distance-time graph does not show the direction of motion of a point at any instant, this direction is usually known and the speed $\frac{ds}{dt}$ is often called the velocity.

4. **Acceleration.**—The rate of change of a velocity is called *acceleration* and it may be uniform or variable. Since velocity involves speed and direction, acceleration involves change of speed or change of direction or change of both speed and direction. Generally, in continuous motion, acceleration is regarded as positive or negative according as the velocity is increasing or decreasing. Negative acceleration is also called *deceleration* or *retardation*. When the direction of motion is reversed at intervals, as, for instance, during vibrating motion, the acceleration is usually taken as positive or negative according as it is in the direction of positive or negative displacement. Acceleration is *linear acceleration* when the variable velocity is linear and it is *angular acceleration* when the variable velocity is angular.

If a point starting from rest moves along a straight line with a uniform acceleration f, then the velocity v after time t is

$$v = ft \quad \text{and} \quad f = v/t.$$

If v is in feet per second and t is in seconds, then f is in feet per second per second. Similarly, for uniform angular acceleration where motion is from rest, the relation between angular velocity ω, angular acceleration α, and time t is

$$\omega = \alpha t \quad \text{or} \quad \alpha = \omega/t.$$

If ω is in radians per second and t is in seconds, then α is in radians per second per second.

The linear acceleration of a falling body whose motion is unresisted is denoted by g, and the value of g is approximately 32·2 feet per second per second. Actually this acceleration, due to gravity, varies slightly at different places on the earth and depends on the altitude, but for engineering purposes it is sufficiently accurate to take it as a constant.

5. Velocity - Time Graphs.—Velocity-time graphs are shown in Figs. 5 and 6. When the velocity increases at a uniform rate (Fig. 5) equal increments δv occur in equal intervals of time δt, the graph is a straight line and the acceleration is $\frac{\delta v}{\delta t}$, which is constant. When the velocity increases at a varying rate (Fig. 6) the increment δv has different values in the equal intervals of time δt. The average acceleration over one of these intervals is $\frac{\delta v}{\delta t}$, and the value to which this ratio approaches if δt is made to approach zero is written $\frac{dv}{dt}$, which is the instantaneous value of the acceleration. When the acceleration is constant, $\frac{dv}{dt}$ has the same value as $\frac{\delta v}{\delta t}$.

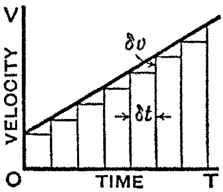

FIG. 5.

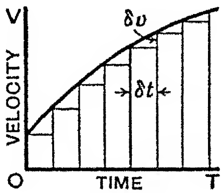

FIG. 6.

The acceleration at any instant may be found graphically by drawing a tangent to the velocity-time graph and measuring the slope, which is the graphical interpretation of $\frac{dv}{dt}$ provided the lengths are measured with the appropriate velocity and time scales. Owing to inaccuracies in drawing and measurement, this graphical method only gives approximate results.

The use of the notation $\frac{dv}{dt}$ for the instantaneous value of the acceleration is exactly the same as the use of $\frac{ds}{dt}$ for the instantaneous value of the velocity which was dealt with in Art. 3. When the equation of the velocity-time graph is known, the value of $\frac{dv}{dt}$ should be found by differentiation. Since $v=\frac{ds}{dt}$, $\frac{dv}{dt}$ may be written $\frac{d\left(\frac{ds}{dt}\right)}{dt}$ or more briefly $\frac{d^2s}{dt^2}$, but it should be understood that $\frac{d^2s}{dt^2}$ is to be taken as one symbol denoting acceleration or rate of change of velocity.

It will be shown later (Arts. 28 and 29) that when motion is along a curve there is an acceleration along the normal at each point, even when the velocity is of uniform magnitude. Therefore, in general, the slope of the velocity-time graph will not give the total acceleration unless the motion is along a straight line.

6. Fluxional Notation for Velocity and Acceleration.—Instead of writing velocity as $\frac{ds}{dt}$ and acceleration as $\frac{dv}{dt}$ or $\frac{d^2s}{dt^2}$, the notation $\dot{s}$ for velocity and $\dot{v}$ or $\ddot{s}$ for acceleration is sometimes very convenient because it is written more quickly and saves space. $\dot{s}$ is read as *s dot* and $\ddot{s}$ is read as *s double dot* or *s two dot*. Similarly, $\dot{\theta}$ denotes angular velocity and $\ddot{\theta}$ or $\dot{\omega}$ denotes angular acceleration.

7. Relations between Linear Motion and Circular Motion.—Let P be any point on a disc which is rotating in its own plane about its centre O and let $OP=r$ (Fig. 7). When OP has turned through θ radians the distance s which P has travelled is given by $s=\theta r$ and at any instant the direction of motion of P is perpendicular to the radius OP.

Since $s=\theta r$, the rate of change of s must be r times the rate of change of θ, therefore, if at any instant v is the linear velocity of P and ω is the angular velocity of OP,

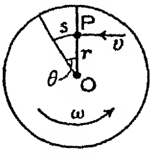

FIG. 7.

$$v=\omega r \quad \text{or} \quad \omega=v/r.$$

For a given value of ω the value of v is proportional to r, for instance if r is doubled then v is also doubled.

At any moment let the acceleration of P tangential to its path be f and let the angular acceleration of OP be a. Since $v=\omega r$, the rate of change of v must be r times the rate of change of ω, therefore

$$f=ar \quad \text{or} \quad a=f/r,$$

and for a given value of a, f is proportional to r.

The units will now be considered. If distance is measured in feet and time in seconds, then since an angle in radians is an arc divided by a radius, or feet divided by feet, or a ratio which cannot have dimensions, therefore

$$\omega=\frac{v}{r}=\frac{\text{feet}}{\text{sec.}}\cdot\frac{1}{\text{feet}}=\frac{\text{radians}}{\text{sec.}},$$

and $$a=\frac{f}{r}=\frac{\text{feet}}{\text{sec.}^2}\cdot\frac{1}{\text{feet}}=\frac{\text{radians}}{\text{sec.}^2}.$$

8. **Vectors.**—Quantities such as displacement, velocity, and acceleration which involve direction as well as magnitude are called *vector quantities* and may be represented by straight lines called *vectors*. For instance, a displacement from A to B (Fig. 8) may be represented by a line *ab* drawn parallel to AB and to some definite scale so that the length *ab* is proportional to the length AB. The directions of the two lines must be the same and it is for this reason that they are made parallel. The *sense* of the direction, that is whether the displacement is from A to B or from B to A, is given by the order in which the letters *a* and *b* are

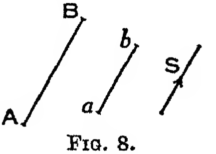

FIG. 8.

mentioned. A displacement from A to B is represented by the vector *ab*, and a displacement from B to A is represented by the vector *ba*. The sense may also be shown by putting an arrowhead on the vector, which may then be labelled with one letter, such as S in the Fig., but with this method the vector can only represent one sense.

Suppose a point P (Fig. 9) is moved in the directions PA and PB simultaneously and it is required to find the resultant displacement. Draw a line *pa* parallel to the direction PA and of such length that it represents to some scale the displacement in the direction PA. Similarly, draw *pb* parallel to PB to represent the displacement in the direction PB. Complete the parallelogram *pacb* and join *pc*, then *pc* represents the resultant displacement of the point P to the same scale that *pa* and *pb* represent the separate displacements. The parallelogram *pacb* is called a *vector parallelogram* or *parallelogram of vectors*. The resultant *pc* is the vector sum of the vectors *pa* and *pb*, and this vector addition may be written as

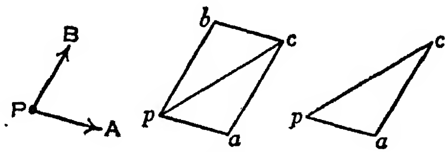

FIG. 9.

$$pa + pb = pc.$$

Various notations are used to ensure that it is understood that such an equation represents the addition of vectors and not an algebraic sum, for instance,

$$\overline{pa} + \overline{pb} = \overline{pc},$$

or
$$\overrightarrow{pa} + \overrightarrow{pb} = \overrightarrow{pc},$$

but in most cases these notations are not essential.

Since *ac* is equal and parallel to *pb*, the vector sum of *pa* and *pb* may also be found by drawing the triangle *pac*, called a *vector triangle*, then

$$pa + pb = pa + ac = pc.$$

The sum of any number of vectors may be found by

drawing a polygon. Suppose it is required to find the resultant of the vectors R, S, T, and U (Fig. 10). Draw *ab* parallel and equal to R, *bc* parallel and equal to S, *cd* parallel and equal to T, and *de* parallel and equal to U. (Of course *ab*, *bc*, etc. may be made proportional to the corresponding vectors R, S, etc.) Join *ae*, then *ae* is the resultant of the given vectors and *abcde* is called a *vector polygon.* Join *ac* and *ad*, then *ac* is the resultant of *ab* and *bc*, *ad* is the resultant of *ac* and *cd*, and *ae* is the resultant of *ad* and *de*, therefore *ae* is the resultant of the given vectors.

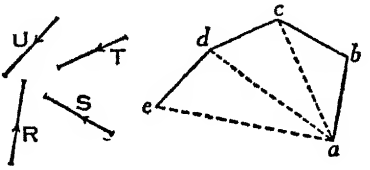

FIG. 10.

So far only displacement vectors have been considered, but velocity is displacement divided by time and therefore a vector triangle or a vector polygon may be drawn in which each vector represents a velocity. Similarly, since acceleration is change of velocity divided by time, a vector triangle or a vector polygon may be drawn in which each vector represents an acceleration.

9. Resolution of Vectors.—Since the vector sum or resultant of two vectors *ab* and *bc* (Fig. 11) may be obtained by drawing a triangle *abc*, conversely the resultant *ac* may be resolved or split up into the two vectors *ab* and *bc*. These two vectors are called *component vectors* or *components* and each may be drawn in any direction as indicated in Fig. 12. It is often desirable to have the components in mutually perpendicular directions (Fig. 13); in this case the fixing of the direction of one component also fixes the direction of the other component.

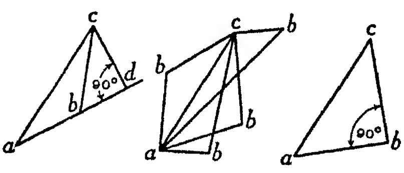

FIG. 11. FIG. 12. FIG. 13.

It will be noticed in Fig. 11 that by drawing *cd* perpen-

dicular to *ab* and meeting *ab* produced at *d*, the component *bc* is resolved into components *bd* and *dc*, *bd* being in the same direction as *ab*; therefore, unless two components are mutually perpendicular, each one may be resolved so as to have a component in the direction of the other.

10. Analytical Determination of Resultants and Components.—Suppose it is required to find by calculation the resultant R of vectors P and Q (Fig. 14) which are inclined to one another at an angle β.

The resultant may be calculated from the cosine formula

$$R=\sqrt{P^2+Q^2-2PQ\cos\beta}.$$

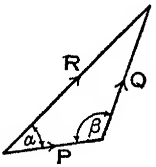

FIG. 14.

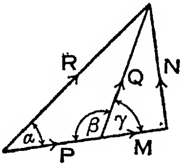

FIG. 15.

Let α be the angle between P and R, then

$$\frac{\sin\alpha}{\sin\beta}=\frac{Q}{R}\quad\text{or}\quad \alpha=\sin^{-1}\left(\frac{Q}{R}\sin\beta\right).$$

The same results could also be obtained without using the cosine formula. Let Q be resolved in two directions, along and perpendicular to the direction of P. Denoting these components by M and N (Fig. 15) and writing γ for $180°-\beta$, then $M=Q\cos\gamma$ and $N=Q\sin\gamma$.

The resultant R is the hypotenuse of a right-angled triangle and is given by

$$R=\sqrt{(P+M)^2+N^2},$$

also $$\sin\alpha=\frac{N}{R}\quad\text{and}\quad \alpha=\sin^{-1}\frac{N}{R},$$

or $$\tan\alpha=\frac{N}{P+M}\quad\text{and}\quad \alpha=\tan^{-1}\frac{N}{P+M}.$$

The second method is better illustrated by the example shown in Fig. 16. Suppose it is required to find the resultant R of vectors P_1, P_2, and P_3, inclined at angles θ_1, θ_2, and θ_3, respectively, to an axis OX.

Draw the axis OY at right angles to OX, then project P_1, P_2, and P_3 on to the axes by drawing the perpendiculars

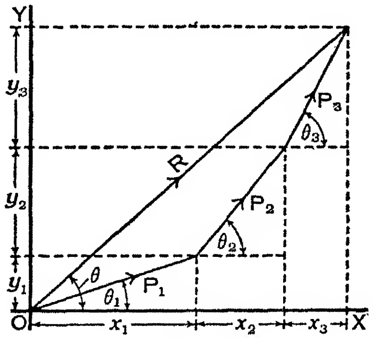

FIG. 16.

as shown by the dotted lines and let the projections be denoted by x_1, y_1, etc.

Then

$$x_1 = P_1 \cos\theta_1, \quad x_2 = P_2 \cos\theta_2, \quad x_3 = P_3 \cos\theta_3,$$
$$y_1 = P_1 \sin\theta_1, \quad y_2 = P_2 \sin\theta_2, \quad y_3 = P_3 \sin\theta_3,$$

and $$R = \sqrt{(x_1+x_2+x_3)^2+(y_1+y_2+y_3)^2}.$$

If R is inclined at an angle θ to OX, then

$$\theta = \sin^{-1}\frac{y_1+y_2+y_3}{R} = \tan^{-1}\frac{y_1+y_2+y_3}{x_1+x_2+x_3}.$$

CHAPTER II

VELOCITY DIAGRAMS—INSTANTANEOUS CENTRES

11. Relative Velocity.—Consider points A and B (Fig. 17) moving in the plane of the paper and having velocities v_a and v_b, respectively, relative to the paper. It is required to find the velocity of B relative to A—that is, the velocity with which B appears to be moving when viewed from A.

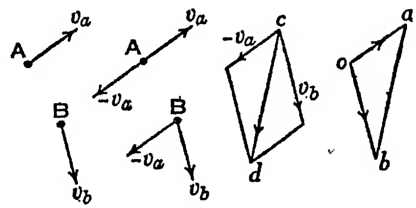

FIG. 17.

Give A and B velocities equal to $-v_a$, then A will be at rest relative to the paper but the velocity of B relative to A will be unaltered. Draw the parallelogram of velocities for B, then the diagonal *cd* represents the velocity of B relative to the paper and therefore relative to A because A is now at rest.

The same result may be obtained more quickly from the vector triangle *oab*, drawing *oa* and *ob* to represent v_a and v_b, respectively, and then joining *ab*. The velocity of B relative to A is represented by *ab*, for it can be seen that $ab = cd$. The arrowheads shown on the triangle *oab* are unnecessary in practice, but they have been put on to emphasize the fact that the vectors *oa* and *ob* represent the *actual velocities* v_a and v_b, neither of these being reversed.

In a similar way, by bringing the point B to rest, it can be shown that the velocity of A relative to B is *ba*.

The point *o*, the starting-point when drawing the triangle, is usually called a *pole*.

In the triangle *oab* there are two ways of getting from

o to b, either direct or via a, and the vector equation representing these two routes is

$$ob = oa + ab.$$

In words—

Velocity of B = *Velocity of* A + *Velocity of* B *relative to* A.

This vector equation may also be written in the form

$$oa = ob - ab,$$

but $$-ab = ba,$$

therefore $$oa = ob + ba.$$

In words—

Velocity of A = *Velocity of* B + *Velocity of* A *relative to* B.

This result also follows immediately from the triangle oab, for there are two ways of getting from o to a, either direct or via b.

12. **Special Cases of Relative Velocity.**—(1) Let A and B be two points on a rigid link moving in the plane of the paper (Fig. 18). The distance from A to B cannot change and so there can be no relative motion between A and B along the line AB. Therefore relative motion must be perpendicular to AB. If rotary motion of the link is clockwise, then relative to B the velocity of A is in the direction Aa perpendicular to AB, and relative to A the velocity of B is in the direction Bb perpendicular to AB.

Fig. 18.

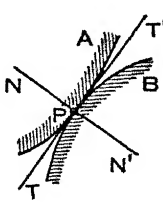

Fig. 19.

(2) Let A and B (Fig. 19) be two rigid links having relative motion in the plane of the paper and being continuously in contact. Let P be the point of contact for an instant, let TPT′ be the common tangent, and let NPN′ be the common normal. Since the links are rigid and remain in contact, there can be no relative motion along the common normal NPN′. Therefore if one link slides on the other, the relative motion at P must be along the common

tangent TPT′. If the motion is pure rolling, then there is no relative motion at P.

13. Angular Velocity Ratios.—Two examples of the determination of angular velocity ratios, in which the results of the preceding Art. are used, will now be given.

Example 1.—Let a link AB turning about a fixed point A with an angular velocity ω_1 (Fig. 20) turn a link CD about a fixed point D with an angular velocity ω_2, the points B and C being coupled by a link BC. The three links all move in the plane of the paper. It is required to find the value of the velocity ratio ω_2/ω_1 when the links are in the positions shown.

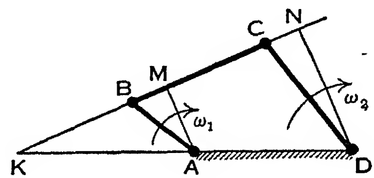

FIG. 20.

Produce CB and DA to intersect at K, and draw AM and DN perpendicular to BC, meeting BC at M and N, respectively.

Assuming that the link BC does not alter in length, the points B and C cannot have any relative velocity along BC. The component of the velocity of B in the direction BC is $\omega_1 . \text{AM}$, and the component of the velocity of C in the direction BC is $\omega_2 . \text{DN}$. Therefore

$$\omega_1 . \text{AM} = \omega_2 . \text{DN} \quad \text{or} \quad \frac{\omega_2}{\omega_1} = \frac{\text{AM}}{\text{DN}}.$$

Since the triangles KAM and KDN are similar,

$$\frac{\text{AM}}{\text{DN}} = \frac{\text{KA}}{\text{KD}}.$$

Therefore
$$\frac{\omega_2}{\omega_1} = \frac{\text{KA}}{\text{KD}}.$$

Example 2.—A cam AB rotating about a fixed point A with an angular velocity ω_1, turns a link DC about a fixed point D with an angular velocity ω_2 (Fig. 21). Assuming that the cam and the link are always in contact, it is

required to find the value of the ratio ω_2/ω_1 for the position shown.

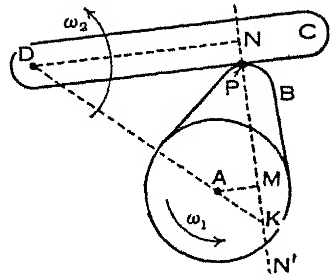

FIG. 21.

Let P be the point of contact. Draw the common normal NPN′ and draw AM and DN perpendicular to it. Draw a line through D and A to intersect NN′ at K.

There cannot be any relative velocity, between the cam and the link, along the common normal NPN′, therefore, as in the preceding example, $\omega_1 . \mathrm{AM} = \omega_2 . \mathrm{DN}$, or

$$\frac{\omega_2}{\omega_1} = \frac{\mathrm{AM}}{\mathrm{DN}} = \frac{\mathrm{KA}}{\mathrm{KD}}.$$

14. Velocities of Points on a Rigid Body—Velocity Image. —Let A and B (Fig. 22) be points on a rigid body having plane motion and let the velocities of the points A and B be v_a and v_b, respectively. Assume that v_a is known in direction and magnitude, but that v_b is known in direction only. It is required to find the magnitude of v_b and then to show how the velocity of any other point on the body may be determined. All points in any line perpendicular to the plane of the paper, which is a plane of motion, have the same velocity, and therefore it is sufficient to consider A and B and other points as being in the plane of the paper.

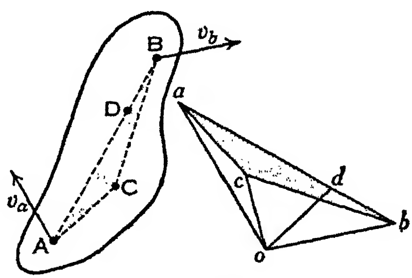

FIG. 22.

The velocity diagram *oab* may now be constructed. To some convenient scale draw *oa* equal and parallel to v_a and draw *ob* parallel to v_b. The length of *ob* is as yet unknown. Since A and B are points on a rigid body, there cannot be relative motion between them along the line AB, therefore

the relative motion between A and B must be in the direction perpendicular to AB. Therefore draw ab perpendicular to AB, intersecting ob at b, then the length ob represents the magnitude of v_b, the velocity of the point B. The vector equation may be written—

Velocity of B = Velocity of A + Velocity of B relative to A,

or $$ob = oa + ab.$$

Similarly, $$oa = ob + ba.$$

Now consider any other point C on the body and in the plane of the paper. Join AC and BC. The velocity of C relative to A is perpendicular to AC and the velocity of C relative to B is perpendicular to BC. Therefore, in the velocity diagram, draw ac and bc perpendicular to AC and BC, respectively, and let c be the point of intersection, then ac represents the velocity of C relative to A and bc represents the velocity of C relative to B. Join oc, then oc represents the velocity of the point C, as can be seen from either of the vector equations

$$oc = oa + ac \quad \text{or} \quad oc = ob + bc.$$

The triangles abc and ABC are similar, since each side of one is perpendicular to a side of the other; the triangle abc is called the *velocity image* of the triangle ABC. The velocity of any point D in the line AB is given by od, where d is found by dividing ab so that $\dfrac{ad}{ab} = \dfrac{\mathrm{AD}}{\mathrm{AB}}$. The angular velocity of the body AB may be found by dividing the velocity of B relative to A (or of A relative to B) by the length AB—that is,

$$\text{Angular velocity of AB} = \frac{ab}{\mathrm{AB}} = \frac{ba}{\mathrm{AB}},$$

and the direction in this particular case is clockwise.

15. Instantaneous Centre.—Another way of determining velocities is by means of the instantaneous centre. Suppose A and B (Fig. 23) are points on a rigid body having plane

motion and that A and B are moving in the plane of the paper. Let v_a and v_b be the velocities of A and B, respectively, when the body is in the given position, and let the directions of these velocities be defined by the angles α and β as indicated. Usually one velocity is known in both magnitude and direction, but the other is known in direction only.

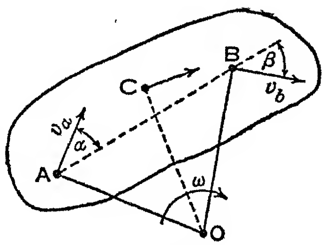

FIG. 23.

Draw AO and BO perpendicular to the directions of the velocities of A and B, respectively, and let AO and BO intersect at O, then O is called the *instantaneous centre* or *virtual centre* of the body relative to the paper.

Since the direction of motion of the point A is perpendicular to AO, the body can turn for an instant about any point in AO without affecting the direction of motion of A; similarly, since the direction of motion of the point B is perpendicular to BO, the body can turn for an instant about any point in BO without affecting the direction of motion of B. Therefore the body can turn about the point O for an instant without affecting the directions of the motions of A and B.

Let ω be the angular velocity of the body, then

$$v_a = \omega . \mathrm{OA}, \quad v_b = \omega . \mathrm{OB}, \quad \text{and by division} \quad \frac{v_a}{v_b} = \frac{\mathrm{OA}}{\mathrm{OB}}.$$

Therefore, if the magnitude of one velocity is known, the magnitude of the other can be determined.

The same result will now be obtained in another way to show that the use of the instantaneous centre is justifiable. Since A and B are points on a rigid body, there cannot be any relative motion between them in the line AB, therefore, resolving along AB,

$$v_a \cos \alpha = v_b \cos \beta, \quad \text{or} \quad \frac{v_a}{v_b} = \frac{\cos \beta}{\cos \alpha}.$$

Now $\cos \alpha = \sin \mathrm{OAB}$ and $\cos \beta = \sin \mathrm{OBA}$,

therefore $$\frac{v_a}{v_b}=\frac{\sin \mathrm{OBA}}{\sin \mathrm{OAB}}=\frac{\mathrm{OA}}{\mathrm{OB}},$$

and this is the relation obtained before, therefore the instantaneous centre method gave the correct result. The velocity of any other point C on the body and in the plane of the paper is ω.OC and its direction is perpendicular to OC.

It will be noticed that OAB is a triangle of velocities turned through a right angle, for OA and OB are perpendicular to the directions of motion of A and B, respectively, and AB is perpendicular to the direction of the relative motion between A and B. The magnitudes of the velocities are represented by the lengths of the sides of the triangle, to the scale on which OA represents the magnitude of the velocity of A.

The method of instantaneous centres is convenient and easy to apply in simple mechanisms, but the velocity diagram method is to be preferred in complex cases. Examples in which both methods are illustrated are given in Arts. 17 and 18.

The instantaneous centre of a moving body is continually changing its position unless the body has rotary motion only, and the locus of the instantaneous centre is called a *centrode*. A line drawn through an instantaneous centre perpendicular to the plane of motion is called an *instantaneous axis*, and the locus of this axis is a surface called an *axode*.

16. Permanent and Fixed Centres.—Two particular cases of instantaneous centres are illustrated in Fig. 24, which shows the slider-crank mechanism. Relative to the crank CB, the connecting-rod AB turns about the point B, and B is the instantaneous centre of AB relative to CB or of CB relative to AB. The point B is also called a *permanent centre* since it always connects AB and CB, although its position is continually changing relative to the paper or the fixed link AC. The point A, connecting the slider to the rod AB, is another permanent centre.

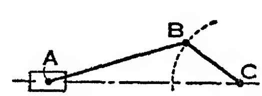

FIG. 24.

The crank CB turns about the fixed point C, and C is the instantaneous centre of CB relative to the paper or the fixed link AC. The point C is also called a *fixed centre* since its position does not change.

17. Velocities in the Slider-Crank Mechanism.—In this Art. a simple example illustrates the use of a velocity diagram and of an instantaneous centre.

A slider at A (Figs. 25 and 26) is connected by a rod AB to the end B of a crank CB of length r. The crank rotates in a clockwise direction about the point C with uniform angular velocity ω, and the slider reciprocates along the line AC. It is required to find the velocity v_a of the slider A and the angular velocity Ω of the rod AB when the crank is in the given position. (The analytical determination of the velocity of the slider is given in Art. 27.)

Let v_b be the velocity of the point B, then $v_b = \omega r$ and its direction is perpendicular to CB.

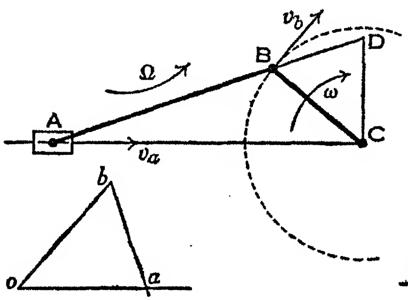

FIG. 25.

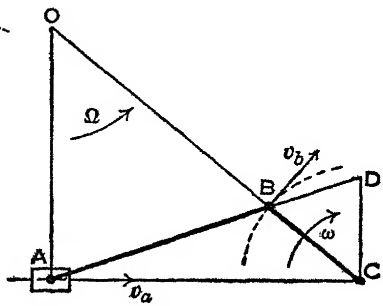

FIG. 26.

Method 1.—*Velocity Diagram.*—From a convenient pole o (Fig. 25) draw ob perpendicular to CB and oa parallel to AC, then, using a suitable scale, make $ob = v_b = \omega r$. Since AB is a rigid link, the velocity of A relative to B is perpendicular to AB, therefore draw ba perpendicular to AB and intersecting oa at a.

Velocity of A = Velocity of B + Velocity of A relative to B,

or $$oa = ob + ba.$$

Therefore oa represents v_a, the velocity of A.

The angular velocity of AB is $\Omega = \frac{ba}{\text{AB}}$, where ba is measured with the velocity scale and AB is measured with the scale to which the mechanism is drawn. The direction of this angular velocity is anticlockwise.

In this particular example the required velocities may also be obtained as follows. Produce AB to intersect at D a line drawn from C perpendicular to AC.

The triangles CBD and oba are similar, since CB, CD, and BD are perpendicular to ob, oa, and ba, respectively,

therefore $$\frac{v_a}{v_b} = \frac{oa}{ob} = \frac{\text{CD}}{\text{CB}},$$

and CD represents the velocity of A to the same scale that CB represents the velocity of B. Also, measuring BD with the same scale and measuring AB with the scale to which the mechanism is drawn, $\Omega = \frac{\text{BD}}{\text{AB}}$.

Since $v_b = \omega . \text{CB}$, using the mechanism scale, therefore $v_a = \omega . \text{CD}$ and $\Omega = \omega \frac{\text{BD}}{\text{AB}}$. The velocities can be found for various positions of the crank by repeating the construction.

Method 2.—*Instantaneous Centre.*—Draw AO (Fig. 26) perpendicular to AC, the direction of motion of A, and produce CB, which is perpendicular to the direction of motion of B, to intersect AO at O. Since A and B are moving perpendicular to OA and OB, respectively, therefore O is the instantaneous centre of AB relative to the paper—that is, relative to AC, for AC is not moving.

Since B is a point in BC which is turning about C, $v_b = \omega . \text{CB}$; also since B is a point in AB which is turning for an instant about O, $v_b = \Omega . \text{OB}$.

Therefore $\Omega . \text{OB} = \omega . \text{CB}$, or $\Omega = \omega \frac{\text{CB}}{\text{OB}}$.

In some cases the point O may be off the paper, therefore produce AB to intersect at D a line drawn from C perpendicular to AC, then OAB and CDB are similar triangles

and $$\frac{CB}{OB}=\frac{BD}{AB}.$$

Therefore $$\Omega=\omega\frac{BD}{AB}.$$

Also $v_a=\Omega . OA=\omega\frac{BD . OA}{AB}$, but $\frac{OA}{AB}=\frac{CD}{BD}$,

therefore $$v_a=\omega . CD,$$

and since $v_b=\omega . CB$, v_a is represented by CD to the same scale that CB represents v_b.

18. Valve Velocity in a Hackworth Valve Gear.—The Hackworth valve gear is shown diagrammatically in Fig. 27. A link CM, rotating anticlockwise with uniform velocity about the fixed centre C, drives a link MN which is pivoted to a slider at N, the slider being constrained by guides to move in a straight line NL. The link MN drives a slider V through a link PV, the slider being constrained by guides to move in a straight line HV. The valve, which is not shown, has the same motion as the slider V.

Given that the linear velocity of the point M is 5 feet per second, it is required to find the velocity of the slider V when the configuration of the mechanism is as shown.

Method 1.—*Velocity Diagram.* — The velocity diagram, shown at (*a*), is constructed as follows. From any pole *o* draw *om* parallel to the direction of motion of the point M, that is perpendicular to CM, making the length *om* repre-

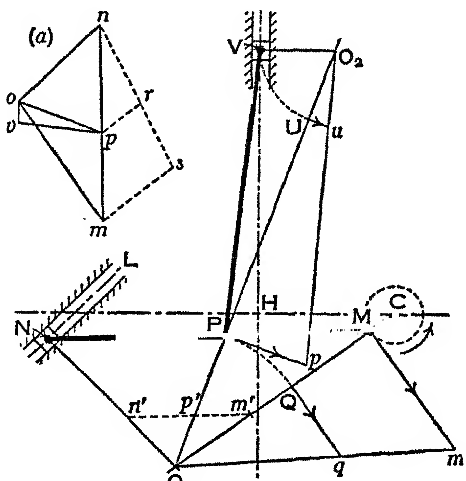

FIG 27.

sent 5 feet per second to some convenient scale. The actual length of *om* is 0·75 inch, but on the original drawing from which Fig. 27 was prepared it was twice as long, and a much larger scale would give greater accuracy. Draw *on* parallel to NL, the direction of motion of N, and draw *mn* perpendicular to MN, meeting *on* at *n*, then *mn* is the velocity image of MN. Divide *mn* at *p* so that $\frac{np}{nm}=\frac{\mathrm{NP}}{\mathrm{NM}}$. This has been done by drawing a line *ns* at a convenient angle to *nm*, making $ns=\frac{1}{2}\mathrm{NM}$ and $nr=\frac{1}{2}\mathrm{NP}$, joining *sm* and drawing *rp* parallel to *sm*. Join *op*, then *op* represents the velocity of P.

Finally, draw *ov* parallel to VH and *pv* perpendicular to PV, meeting *ov* at *v*, then *ov* represents the velocity of the point V in direction and magnitude. By measurement $ov=0{\cdot}11$ inch, therefore the velocity of V is $\frac{5}{0{\cdot}75}\times 0{\cdot}11=0{\cdot}73$ foot per second, approximately.

Method 2.—*Instantaneous Centres*.—Produce CM to intersect at O_1 a line drawn from N perpendicular to NL, then O_1 is the instantaneous centre of the link MN. Join $\mathrm{O}_1\mathrm{P}$, then P is moving perpendicular to $\mathrm{O}_1\mathrm{P}$. Produce $\mathrm{O}_1\mathrm{P}$ to intersect at O_2 a line drawn from V perpendicular to HV, then O_2 is the instantaneous centre of the link PV. To some scale, $\mathrm{O}_1\mathrm{M}$, $\mathrm{O}_1\mathrm{P}$, and $\mathrm{O}_1\mathrm{N}$ represent the velocities of M, P, and N, respectively, and to some other scale, $\mathrm{O}_2\mathrm{P}$ and $\mathrm{O}_2\mathrm{V}$ represent the velocities of P and V, respectively, but it is unnecessary to know these scales to find the velocity of the slider V.

Velocity of $\mathrm{V}=5\times\frac{\mathrm{O}_1\mathrm{P}}{\mathrm{O}_1\mathrm{M}}\times\frac{\mathrm{O}_2\mathrm{V}}{\mathrm{O}_2\mathrm{P}}$ feet per second, and if the lengths $\mathrm{O}_1\mathrm{P}$, etc., are measured, then the numerical value of the velocity of V may be calculated, but a graphical solution is given in the Fig.

Draw M*m* perpendicular to $\mathrm{O}_1\mathrm{M}$ and equal to 0·75 inch to represent the velocity of M—that is, 5 feet per second. As previously stated, a larger scale would give greater accuracy. Join O_1m. With centre O_1 and radius $\mathrm{O}_1\mathrm{P}$

draw an arc PQ meeting O_1M at Q, then draw Qq parallel to Mm and meeting O_1m at q. Draw Pp perpendicular to O_1P and equal to Qq, then Pp represents the velocity of P in direction and magnitude.

Join O_2p. With centre O_2 and radius O_2V draw an arc VU meeting O_2P at U, then draw Uu parallel to Pp and meeting O_2p at u. The magnitude of the velocity of V is represented by Uu and its true direction is along VH. The proof will be left to the student. By measurement $Uu=0{\cdot}11$ inch, therefore the velocity of V is $\dfrac{5}{0{\cdot}75}\times 0{\cdot}11$ $=0{\cdot}73$ foot per second, approximately, as before. If it happens that an instantaneous centre is off the paper and cannot be used, the difficulty may be overcome graphically. Suppose, for instance, the point O_1 is inaccessible and the velocity of the point P is wanted. Along MO_1 mark off Mm' equal to Mm. Draw $m'n'$ parallel to MN and meeting NO_1 at n'. Divide $m'n'$ at p' so that $\dfrac{n'p'}{n'm'}=\dfrac{NP}{NM}$ and join $p'P$. The construction for obtaining the point p' would be similar to that shown by the dotted lines in the velocity diagram at (a).

The velocities of P and M are proportional to their distances from O_1, also, since $m'n'$ is parallel to MN,

$$\frac{O_1P}{O_1M}=\frac{Pp'}{Mm'},$$

therefore
$$\frac{\text{Velocity of P}}{\text{Velocity of M}}=\frac{O_1P}{O_1M}=\frac{Pp'}{Mm'}.$$

Therefore Pp' is the magnitude of the velocity of P to the same scale that Mm' is the magnitude of the velocity of M. The direction of the velocity of P is perpendicular to Pp', clockwise about O_1, since M is moving clockwise about O_1. Similarly, to the same scale, Nn' is the magnitude of the velocity of N and the direction is perpendicular to Nn', clockwise about O_1.

19. Relative Instantaneous Centres of any Three Links in a Plane.—Let the thick lines A, B, and C (Fig. 28) represent

any three links which are in the same plane, and let O_{AB}, O_{BC}, and O_{AC} be the three instantaneous centres for relative motion between the links. For instance, when there is relative motion between A and B, then, for an instant, either link turns relative to the other about the point O_{AB}. Similarly for B and C and for A and C. It will now be proved that the three instantaneous centres lie in one straight line.

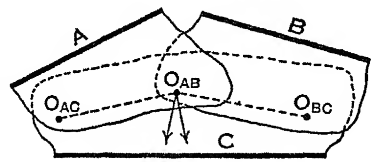

FIG. 28.

Since the links as drawn do not overlap, it may be helpful to imagine a separate piece of paper attached to each in order to overlap at the instantaneous centres as indicated in the Fig.

As B moves relative to C, it turns for an instant about O_{BC}, and the point O_{AB}, considered as a point on B, moves perpendicular to the straight line $O_{AB}O_{BC}$. Also, as A moves relative to C it turns for an instant about O_{AC}, and the point O_{AB}, considered as a point on A, moves perpendicular to the straight line $O_{AC}O_{AB}$. Now the relative motion between A and B is one of rotation about O_{AB}, therefore this point can move only in one direction relative to C, and since this direction is perpendicular to $O_{AB}O_{BC}$ and to $O_{AC}O_{AB}$, it follows that $O_{AB}O_{BC}$ and $O_{AC}O_{AB}$ must be in one straight line. Therefore the instantaneous centres O_{AC}, O_{AB}, and O_{BC} lie in one straight line.

Example.—Consider the mechanism known as a *four-bar chain*, consisting of four connected links and labelled, 1, 2, 3, and 4 in Fig. 29. Any one of these links may be fixed, leaving the other three links free to move.

The various instantaneous centres will now be determined. Relative motion between links 1 and 2 can only occur by rotation about the point labelled O_{12}, and this is the

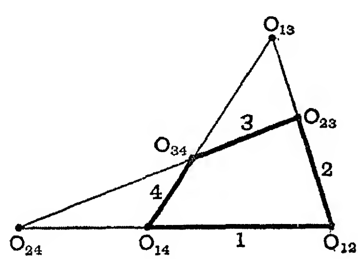

FIG. 29.

instantaneous centre for link 1 relative to link 2 or *vice versa.* Similarly, the instantaneous centres O_{23}, O_{34}, and O_{14} are the points so indicated. Now, suppose link 1 is fixed, then the directions of motion of the end points of link 3 are known—that is, the point O_{23} may move at right angles to link 2 and the point O_{34} may move at right angles to link 4, therefore O_{13} is at the intersection of links 2 and 4 produced, and the link 3 may turn for an instant about the instantaneous centre O_{13}. The same point would be arrived at if link 3 were supposed fixed and the motion of link 1 were considered. The point O_{24} is obtained in a similar manner.

Now consider any three of the four links and it will be seen that their instantaneous centres lie in a straight line. For instance, the instantaneous centres of the links 1, 2, and 3 lie in the straight line $O_{12}O_{23}O_{13}$.

20. Centrodes.—As defined in Art. 15, a centrode is the locus of an instantaneous centre. It will now be demonstrated that if any two links of a mechanism move in one plane then their relative motion may be produced by rolling together two centrodes fixed to the links.

Let A_1B_1, A_2B_2, A_3B_3, and A_4B_4 be four positions of a

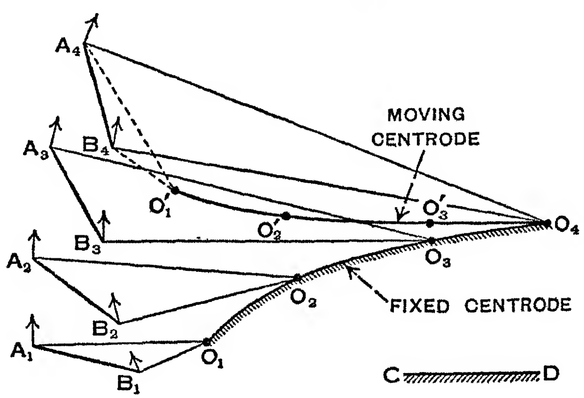

FIG. 30.

moving link AB (Fig. 30) relative to a link CD which will be taken as fixed, and let the points A and B be moving in

the directions shown by the arrows when AB is in each of its four positions.

Draw A_1O_1 and B_1O_1 perpendicular to the directions of motion of A_1 and B_1, respectively, then the point O_1 at the intersection of A_1O_1 and B_1O_1 is the instantaneous centre for the position A_1B_1. The instantaneous centres O_2, O_3, and O_4 are found in a similar way. Now draw a smooth curve through these points. If a great many positions of the moving link were considered, then the curve could be drawn fairly accurately. This curve is the locus of the instantaneous centre of the moving link, therefore it is a centrode and since it is fixed relative to the fixed link CD it may be called the *fixed centrode.*

Suppose now that the moving link AB had a piece of tracing paper attached to it when in its first position. The instantaneous centres could be marked on the tracing paper as it moved with the link and a centrode could be drawn through these points. This centrode may be called the *moving centrode.*

The moving centrode can be drawn in one or more positions without using the tracing paper. For instance, for the position when AB is at A_4B_4, begin by drawing the triangle $A_4B_4O_1'$ equal to the triangle $A_1B_1O_1$. It is clear that if the moving link returns to its first position, taking the triangle $A_4B_4O_1'$ with it, then the point O_1' would coincide with the point O_1. By drawing two more triangles (not shown in the Fig.) the points O_2' and O_3' can be found, then a smooth curve through the points O_1', O_2', O_3', and O_4 gives one position of the moving centrode.

Every point on the moving centrode is in turn coincident with a point on the fixed centrode, and the moving link turns for an instant about each of these coincident points. Therefore the moving centrode rolls on the fixed centrode, and the relative motion between AB and CD may be obtained by rolling the one curve on the other.

The same relative motion between the two links is obtained if the link AB and the centrode $O_1'O_2'O_3'O_4$ are fixed and the centrode $O_1O_2O_3O_4$ is allowed to roll taking with it the link CD.

The student is advised to draw his own figure to a large scale, then to obtain the moving centrode on a sheet of tracing paper and roll one centrode on the other. It is worth doing.

21. Centrodes for Double Slider Mechanism.—The link AB (Fig. 31) is pivoted at its ends to sliders which are constrained so that they can move along fixed straight lines DD′ and EE′ intersecting at C. It is required to find the centrodes for the relative motion between the link AB and the straight lines DD′ and EE′ which will be designated simply as the link DCE.

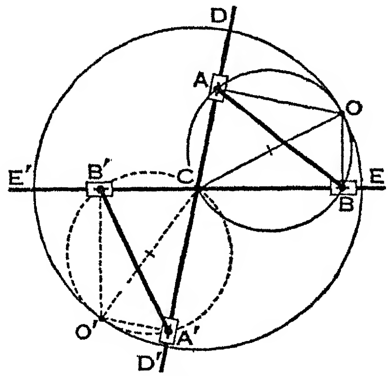

Fig. 31.

Draw AO perpendicular to CD, and BO perpendicular to CE, letting AO and BO intersect at O, then O is the instantaneous centre for the link AB when it is in the position shown.

Now suppose the link AB to be fixed and the link DCE to be free, in order to find how the point O moves relative to AB. Since the angles CAO and CBO are right angles and since the angle ACB is constant and the length AB is constant, therefore the points C, A, O, and B lie on a circle and the locus of O is a circle drawn with CO as a diameter.

To find how the point O moves relative to the link DCE, fix DCE and let AB be free to move. Since CO is of constant length, the locus of O is a circle drawn with its centre at C and with radius equal to CO.

The length CO may be expressed in terms of AB and the angle ACB. Since CAOB is a cyclic quadrilateral, the angle COB is equal to the angle CAB, therefore

$$CO = \frac{CB}{\sin COB} = \frac{CB}{\sin CAB} = \frac{AB}{\sin ACB}.$$

The motion of the link AB relative to the link DCE may be obtained by fixing it to the circle CAOB and rolling this circle, without slipping, on the inside of the fixed circle DOEO′. The moving link AB is shown in another position at A′B′.

Exercises II

1. The points A and B (Fig. 32) are moving in the directions shown. The speed of A is 20 feet per second and the speed of B is 40 feet per second. Find the velocity of B relative to A.

2. If in the preceding exercise the given velocity of B is the velocity of B relative to A, find the true velocity of B.

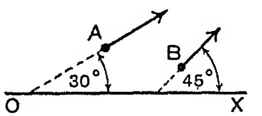

Fig. 32.

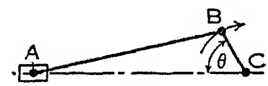

Fig. 33.

3. In the slider-crank mechanism (Fig. 33) $AB=4$ feet, $BC=1$ foot, and the velocity of the crank pin B is 10 feet per second. Find the velocity of the slider A when the crank angle θ has the values 30°, 45°, 60°, and 90°.

4. State the position of the instantaneous centre of the connecting-rod AB (Fig. 33), (*a*) when $\theta=0°$, (*b*) when $\theta=90°$.

5. The line of stroke of a piston A (Fig. 34) is offset a perpendicular distance of $\frac{1}{2}$ inch from the centre C of the crankshaft. The crank CB is rotating clockwise at 1500 revolutions per minute, $CB=2\frac{1}{2}$ inches, and $AB=11$ inches. Find the velocity of the piston A in feet per second when the angle $DCB=105°$, DC being parallel to the line of stroke.

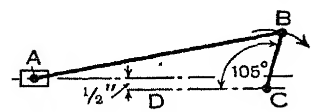

Fig. 34.

6. In the preceding exercise, assume that the outside diameters of the crankshaft journal and the crank pin B are 2 inches and $1\frac{3}{4}$ inches, respectively. Find the rubbing speed, in feet per second, of each journal when the crank is in the given position.

7. (*a*) A locomotive driving wheel rolls along a rail without slipping. Where is the instantaneous centre and what is the velocity of the top point of the wheel if the middle point is moving at 80 feet per second?

(*b*) The wheel turns and slips on the rail when the locomotive is at rest. Where is the instantaneous centre of the wheel?

(*c*) The wheel is 6 feet in diameter and a point on the rim has a velocity of 12 feet per second relative to the locomotive, which

is moving at 5 feet per second. Where is the instantaneous centre of the wheel, what is the sliding velocity of the bottom point, and what is the velocity of the top point?

(*d*) Using the instantaneous centre, find the velocity of the foremost point of the wheel when the latter is moving as described in (*c*).

8. A water turbine blade is shown in Fig. 35. The edge A is moving at 50 feet per second in the direction Aa, and the edge B is moving at 58 feet per second in the direction Bb. The tangent to the blade at A makes an angle ϕ with Aa, and the tangent at B makes an angle of 15° with Bb. Water enters at A without shock and with a velocity of 100 feet per second, inclined at 20° to Aa, as shown, and leaves at B with a velocity v inclined at an angle θ to Bb. The velocity of the water relative to the blade is the same at B as at A.

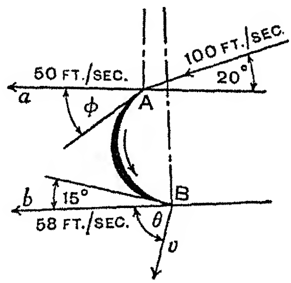

FIG. 35.

Find (*a*) the angle ϕ, (*b*) the velocity of the water relative to the blade, (*c*) the velocity v, and (*d*) the angle θ.

9. Fig. 36 shows the mechanism of the Robinson Air Engine. The crank OC is $2\frac{3}{4}$ inches long. Connecting-rod PC is 12 inches long. BC = 5 inches, O′B = 12 inches, BE = 4 inches, EF = $6\frac{1}{2}$ inches.

Draw the velocity diagram for the mechanism when the crank angle $\theta = 30°$ and the crank OC rotates at 180 revolutions per minute. State the velocities of the two pistons in feet per second. [U.L.]

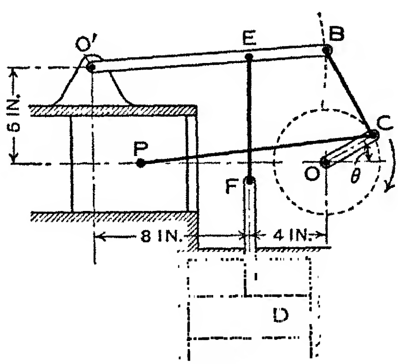

FIG. 36.

10. The link AD of the four-bar chain ABCD (Fig. 37) is fixed and the link AB is rotating clockwise at 20 revolutions per minute. The lengths of the links are as indicated. For the configuration shown, where the angle BAD = 135°, find the angular velocity in radians per second of BC and of CD, the

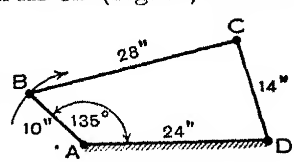

FIG. 37.

velocity of the point C, and the velocity of the middle point of BC.

11. In the mechanism shown in Fig. 38 the crank BC is connected by links to three crossheads A, F, and H. The line of stroke of A passes through the centre C, and F and H move parallel to AC. If BC is rotating clockwise at 150 revolutions per minute, find the velocities of A, F, and H when the angle ACB is 120° as shown. The dimensions are as follows: BC = 8 inches, AB = 28 inches, BE = 31 inches, AE = 13 inches, DE = 26 inches and is one straight link, DF = 8·5 inches, and EH = 8·1 inches. The lines of stroke of F and H are at perpendicular distances of 13·6 inches from the line AC.

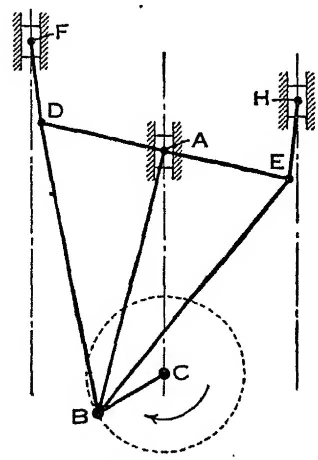

Fig. 38.

12. Referring to the diagram of the Hackworth valve gear (Fig. 27), find the velocity of the slider V when the angle HCM is 90° and the point M has a velocity of 4 feet per second. The other particulars are to be taken as follows: VH is perpendicular to CH and CH = 14 inches, NL is inclined at 45° to CH and intersects CH produced at 33 inches from C, CM = 3 inches, MN = 33½ inches, MP = 15 inches, and PV = 30 inches.

13. Fig. 39 is a diagram of the mechanism of a valve gear for a certain setting, the lengths of the links which are not given in

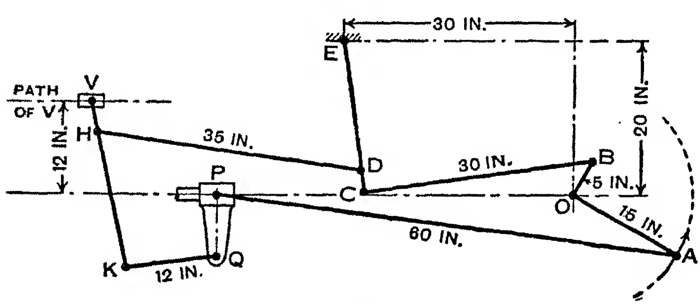

Fig. 39.

the figure being as follows: VH = 4 inches, VK = 22 inches, PQ = 8 inches, ED = 17 inches, EC = 20 inches.

The cranks OA and OB rotate together, being 90° apart, and the velocity of the main crank-pin A is 30 feet per second. Draw

a velocity diagram and determine the velocity of the valve spindle when the crank angle AOP is 150°. (Scales—Draw the mechanism $\frac{1}{8}$th full size; 1 inch = 5 feet per second.) [C.U.]

14. Four links AB, AC, BD, and DC are such that AB = DC and AC = BD. They are jointed at A, B, C, and D in such a way that the links AC and BD cross each other. Prove that the relative motion of AB and CD is the same as that obtained by the rolling of two equal ellipses on one another with foci A and B and C and D, respectively.

Further, show that if the ellipses are free to rotate about A and C, respectively, and one is given a uniform angular velocity, the fractional fluctuation of speed of the other is given by $\frac{4a}{1-a^2}$ where $a = \frac{AB}{AC}$. [C.U.]

CHAPTER III

ANALYSIS OF VELOCITY AND ACCELERATION

22. An Application of Differentiation and Integration.—Denoting displacement by s and time by t, then velocity, or the rate of change of displacement, is given in magnitude by

$$v=\frac{ds}{dt},$$

the direction being the same as the direction of motion.

It is understood that the direction of motion is known whenever the term velocity is used. If the direction is unknown, then, strictly speaking, the term speed should be used, because velocity involves direction as well as magnitude.

Acceleration or the rate of change of velocity, *in the direction of motion,* is given by

$$f=\frac{dv}{dt}=\frac{d^2s}{dt^2}.$$

It will be seen later (Arts. 28 and 29) that when the motion is along a curve there is an acceleration perpendicular to the direction of motion.

If displacement is completely defined by one or more expressions, then the velocity and acceleration may be obtained by differentiation. Conversely, the velocity and displacement may be found from the acceleration by the process of integration.

Example 1.—The displacement s of a body along a straight line is given by the relation $s=4t^2-9$. It is required to find expressions for the velocity and the acceleration.

Since $$s=4t^2-9,$$

$$\text{velocity } v=\frac{ds}{dt}=8t,$$

and
$$\text{acceleration } f=\frac{dv}{dt}=\frac{d^2s}{dt^2}=8.$$

If s is measured in feet and t in seconds, then the velocity is in feet per second and the acceleration is in feet per second per second.

Example 2.—The acceleration of a body having rectilinear motion is 8 ft./sec.². It is required to find the velocity and the displacement in terms of time t, given that the velocity is 16 ft./sec. when $t=2$ sec. and that the displacement is zero when $t=1{\cdot}5$ sec.

Since
$$\frac{d^2s}{dt^2}=8,$$

integrating,
$$v=\frac{ds}{dt}=8t+\mathrm{A},$$

where A is an arbitrary constant or constant of integration.

Now $v=16$ when $t=2$ and substitution in the velocity equation gives

$$16=8\times 2+\mathrm{A}, \quad \text{from which} \quad \mathrm{A}=0,$$

therefore
$$v=\frac{ds}{dt}=8t,$$

where t is in sec. and v is in ft./sec.

Integrating again, $s=4t^2+\mathrm{B}$,

where B is a constant of integration.

Now $s=0$ when $t=1{\cdot}5$ and substitution in the displacement equation gives

$$0=4\times 1{\cdot}5^2+\mathrm{B}, \quad \text{from which} \quad \mathrm{B}=-9,$$

therefore
$$s=4t^2-9,$$

where t is in sec. and s is in ft.

It should be noticed that the two integrations produce

two arbitrary constants and that these constants cannot be evaluated unless two conditions are known. In this example the known conditions are the velocity and the displacement at certain times.

23. Alternative Expressions for Acceleration.—In the preceding Art. acceleration has been written as

$$f=\frac{d^2s}{dt^2}=\frac{dv}{dt} \quad . \quad . \quad . \quad (1).$$

Now
$$\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}=v\frac{dv}{ds},$$

so that
$$f=v\frac{dv}{ds} \quad . \quad . \quad . \quad . \quad (2),$$

is another way of expressing acceleration.

In (1) the independent variable is time t, and in (2) it is displacement s.

Separating the variables and integrating between appropriate limits, say v_1, v_2, t_1, t_2, s_1, and s_2, then from (1)

$$\int_{v_1}^{v_2} dv=\int_{t_1}^{t_2} fdt,$$

and from (2),
$$\int_{v_1}^{v_2} vdv=\int_{s_1}^{s_2} fds.$$

Both these forms are used in the next Art.

24. Uniform Acceleration—Formulæ.—Suppose that a body moving in a straight line with uniform acceleration f has an initial velocity u and that at time t the displacement is s and the velocity is v, both t and s being zero when $v=u$. It is required to find the relations between f, v, u, t, and s.

Since
$$f=\frac{dv}{dt},$$

therefore
$$\int_{u}^{v} dv=\int_{0}^{t} fdt.$$

Integrating, $$v-u=ft,$$

or $$v=u+ft \quad . \quad . \quad . \quad . \quad (1).$$

Since $$f=\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}=v\frac{dv}{ds},$$

therefore $$\int_u^v vdv=\int_0^s fds.$$

Integrating, $$\left[\tfrac{1}{2}v^2\right]_u^v=fs,$$

$$\tfrac{1}{2}(v^2-u^2)=fs,$$

or $$v^2=u^2+2fs \quad . \quad . \quad . \quad (2).$$

From (1) and (2), eliminating f,

$$s=\tfrac{1}{2}(u+v)t \quad . \quad . \quad . \quad (3).$$

From (1) and (3), or from (1) and (2), eliminating v,

$$s=ut+\tfrac{1}{2}ft^2 \quad . \quad . \quad . \quad (4).$$

If the initial velocity u is zero, then equations (1), (2), (3), and (4) become

$$v=ft, \quad v^2=2fs, \quad s=\tfrac{1}{2}vt, \quad \text{and} \quad s=\tfrac{1}{2}ft^2 \quad . \quad (5),$$

respectively.

Similar formulæ may be obtained for uniformly accelerated rotary motion.

Example.—A vehicle travels from A to B, a distance of 400 yards. Starting from rest at A, the vehicle is uniformly accelerated for 16 seconds to a speed of 24 miles per hour. It travels at this speed until it is 64 yards from B, then it is uniformly retarded and brought to rest at B. It is required to find the time taken to travel from A to B, the acceleration and the retardation.

Foot and second units will be used. The uniform speed is $v=24\times\frac{88}{60}=35\cdot2$ ft./sec. Let t_1, t_2, and t_3 be the times of acceleration, uniform speed, and retardation respectively, and let s_1, s_2, and s_3 be the corresponding distances. Also let f_1 be the acceleration and f_3 the retardation.

The velocity-time graph for the journey is shown in

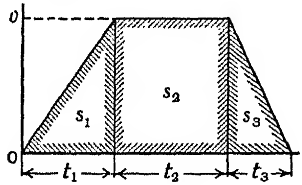

FIG. 40.

Fig. 40, the shaded areas representing the distances as indicated.

$$t_1 = 16 \text{ sec.} \quad s_1 = \tfrac{1}{2}vt_1 = \tfrac{1}{2} \times 35{\cdot}2 \times 16 = 281{\cdot}6 \text{ feet.}$$

Since $\qquad s_1 + s_2 + s_3 = 400 \times 3 = 1200$ feet

and $s_3 = 64 \times 3 = 192$ feet, therefore

$$s_2 = 1200 - s_1 - s_3 = 1200 - 281{\cdot}6 - 192 = 726{\cdot}4 \text{ feet.}$$

Now $s_2 = vt_2$, therefore $t_2 = \dfrac{s_2}{v} = \dfrac{726{\cdot}4}{35{\cdot}2} = 20{\cdot}64$ sec.

Also $s_3 = \tfrac{1}{2}vt_3$, therefore $t_3 = \dfrac{2s_3}{v} = \dfrac{2 \times 192}{35{\cdot}2} = 10{\cdot}91$ sec.

Total time is $t_1 + t_2 + t_3 = 16 + 20{\cdot}64 + 10{\cdot}91 = 47{\cdot}55$ sec.

$$\text{Acceleration } f_1 = \frac{v}{t_1} = \frac{35{\cdot}2}{16} = 2{\cdot}20 \text{ ft./sec.}^2.$$

$$\text{Retardation } f_3 = \frac{v}{t_3} = \frac{35{\cdot}2}{10{\cdot}91} = 3{\cdot}23 \text{ ft./sec.}^2.$$

25. Resultant Velocity and Resultant Acceleration.—If at time t a particle P has displacements x and y (Fig. 41) measured parallel to mutually perpendicular axes OX and OY respectively, then its velocities are

$$\dot{x} \quad \text{and} \quad \dot{y},$$

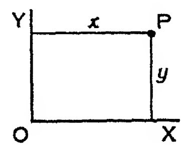

FIG. 41.

using the fluxional notation to save space (see Art. 6, p. 6).

The accelerations of P are

$$\ddot{x} \quad \text{and} \quad \ddot{y}.$$

The resultant velocity at time t is (Fig. 42)

$$V=(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}},$$

in a direction making an angle ϕ with the x axis, and

$$\tan\phi=\dot{y}/\dot{x}.$$

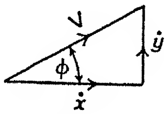

FIG. 42.

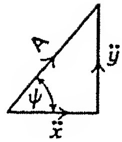

FIG. 43.

The resultant acceleration at time t is (Fig. 43)

$$A=(\ddot{x}^2+\ddot{y}^2)^{\frac{1}{2}}$$

in a direction making an angle ψ with the x axis, and

$$\tan\psi=\ddot{y}/\ddot{x}.$$

It is to be noted that the resultant acceleration *cannot*, in general, be obtained by differentiating the resultant velocity.

Example 1.—A particle P (Fig. 44) is moving in a circular path of radius OP $=r$ with uniform speed v. It is required to investigate the motion and to find the acceleration of the particle by considering the x and y components of the displacement.

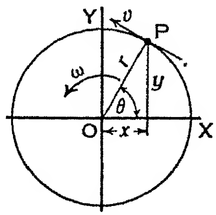

FIG. 44.

In Art. 28, p. 47, it is shown by using vectors that the acceleration is v^2/r or $\omega^2 r$, where ω is the angular velocity of the radius OP, and the direction of the acceleration is along the radius OP from P towards O. This result is now to be found analytically. The vector method is very much simpler in this case, but the analytical method is instructive.

Let OP make an angle θ with OX at time t, then from the Fig. it can be seen that

$$x=r\cos\theta,\quad y=r\sin\theta.$$

Now $\dfrac{dx}{dt}=\dfrac{dx}{d\theta}\dfrac{d\theta}{dt}$ or $\dot{x}=\dfrac{dx}{d\theta}\dot{\theta}$ and similarly $\dot{y}=\dfrac{dy}{d\theta}\dot{\theta}$, therefore, differentiating with respect to t, the velocities are

$$\dot{x}=-r\sin\theta\,\dot{\theta},\quad \dot{y}=r\cos\theta\,\dot{\theta}.$$

Differentiating again with respect to t, the accelerations are

$$\ddot{x} = -r \cos \theta \, \dot{\theta}^2, \quad \ddot{y} = -r \sin \theta \, \dot{\theta}^2.$$

The resultant velocity is

$$(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}} = (r^2 \sin^2 \theta \, \dot{\theta}^2 + r^2 \cos^2 \theta \, \dot{\theta}^2)^{\frac{1}{2}} = r\dot{\theta} = r\omega = v,$$

which is obviously correct.

If the direction of this velocity is inclined at an angle ϕ to the axis OX, then

$$\tan \phi = \frac{\dot{y}}{\dot{x}} = \frac{r \cos \theta \, \dot{\theta}}{-r \sin \theta \, \dot{\theta}} = -\cot \theta,$$

and a little consideration will show that this direction is perpendicular to the radius OP.

The resultant acceleration is

$$(\ddot{x}^2 + \ddot{y}^2)^{\frac{1}{2}} = (r^2 \cos^2 \theta \, \dot{\theta}^4 + r^2 \sin^2 \theta \, \dot{\theta}^4)^{\frac{1}{2}} = r\dot{\theta}^2 = r\omega^2 = v^2/r.$$

If the direction of this acceleration is inclined at an angle ψ to the axis OX, then

$$\tan \psi = \frac{\ddot{y}}{\ddot{x}} = \frac{-r \sin \theta \, \dot{\theta}^2}{-r \cos \theta \, \dot{\theta}^2} = \tan \theta,$$

from which $\psi = \theta$ or $\theta + 180°$. Since the two components of the acceleration are negative, the second of these angles is the one required. Therefore the resultant acceleration of P is along the radius OP and is directed from P towards O.

It is evident that differentiating the resultant velocity v would not produce the resultant acceleration; it would merely show that the component of the acceleration in the direction of the resultant velocity is zero.

Example 2.—The x and y components of the displacement of a body at time t are given by

$$x = 2t^2 + 3 \quad \text{and} \quad y = t^3 - 4t^2 - 2,$$

using foot and second units. It is required to find the resultant velocity and resultant acceleration when $t = 5$ seconds.

Since $x=2t^2+3$ and $y=t^3-4t^2-2$,

therefore $\dot{x}=4t$, $\dot{y}=3t^2-8t$,

and $\ddot{x}=4$, $\ddot{y}=6t-8$.

When $t=5$, $\dot{x}=4\times5=20$ and $\dot{y}=3\times5^2-8\times5=35$.

Resultant velocity is $(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}=(20^2+35^2)^{\frac{1}{2}}=40{\cdot}31$ ft./sec. If ϕ is the angle the direction of this velocity makes with the x axis,

$$\tan\phi=\tfrac{35}{20}=1{\cdot}75 \quad \text{and} \quad \phi=60^\circ\ 15'.$$

When $t=5$, $\ddot{x}=4$ and $\ddot{y}=6\times5-8=22$.

Resultant acceleration is $(\ddot{x}^2+\ddot{y}^2)^{\frac{1}{2}}=(4^2+22^2)^{\frac{1}{2}}=22{\cdot}36$ ft./sec.². If ψ is the angle the direction of this acceleration makes with the x axis,

$$\tan\psi=\tfrac{22}{4}=5{\cdot}5 \quad \text{and} \quad \psi=79^\circ\ 42'.$$

26. Motion of a Projectile.—A shell is fired from a gun with an initial velocity u and at an angle a to the horizontal (Fig. 45). Neglecting air resistance, it is required to find the time of flight T, the range R, the greatest height H, and the equation of the trajectory.

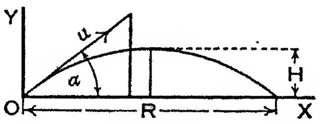

FIG. 45.

The horizontal velocity is constant and, due to gravity, there is a downward acceleration g, or, taking the upward direction as positive, the acceleration is $-g$. The axes are OX and OY as shown.

Resolving the initial velocity horizontally and vertically, the components are $u\cos a$ and $u\sin a$.

Since the horizontal velocity is constant, the horizontal distance travelled in time t is

$$x=ut\cos a \quad . \quad . \quad . \quad (1).$$

Since $$\frac{d^2y}{dt^2}=-g,$$

integration gives $$\frac{dy}{dt}=-gt+\text{A},$$

where A is a constant of integration.

When $t=0$, $\frac{dy}{dt}=u\sin\alpha$, therefore $A=u\sin\alpha$,

and
$$\frac{dy}{dt}=u\sin\alpha-gt.$$

Integrating, the vertical distance travelled in time t is

$$y=ut\sin\alpha-\tfrac{1}{2}gt^2+B,$$

where B is a constant of integration.

When $t=0$, $y=0$, therefore $B=0$, and

$$y=ut\sin\alpha-\tfrac{1}{2}gt^2 \quad . \quad . \quad . \quad (2).$$

When the shell strikes the ground, $y=0$, therefore the time of flight T is obtained by putting $y=0$ in (2) and solving for t.

Therefore
$$t(u\sin\alpha-\tfrac{1}{2}gt)=0,$$

from which
$$t=0 \quad \text{or} \quad t=\frac{2u\sin\alpha}{g}.$$

When $t=0$, the shell is in the gun and the second value is the one required, therefore

$$T=\frac{2u\sin\alpha}{g} \quad . \quad . \quad . \quad (3).$$

Substituting this value of t in (1) gives the range $x=R$, that is

$$R=u\frac{2u\sin\alpha}{g}\cos\alpha=\frac{u^2\sin 2\alpha}{g} \quad . \quad . \quad (4).$$

The greatest height H is the value of y in (2) when $t=\tfrac{1}{2}T=\frac{u\sin\alpha}{g}$, that is

$$H=u\frac{u\sin\alpha}{g}\sin\alpha-\tfrac{1}{2}g\frac{u^2\sin^2\alpha}{g^2}=\frac{u^2\sin^2\alpha}{2g} \quad . \quad (5).$$

The equation of the trajectory is found by eliminating t from (1) and (2), then

$$y=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha} \quad . \quad . \quad (6),$$

which is the equation of a parabola.

27. Expressions for the Approximate Velocity and Acceleration in the Slider-Crank Mechanism.—Let r be the length of the crank CB (Fig. 46), l the length of the connecting-rod AB, and ω the uniform angular velocity of the crank. At time t let x be the displacement of the slider A from the beginning of its stroke, and let θ and ϕ be the angles turned through by the crank and connecting-rod respectively.

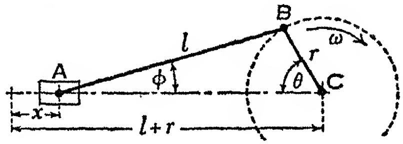

Fig. 46.

Displacement $x = l + r - l\cos\phi - r\cos\theta.$

Now $$l\sin\phi = r\sin\theta \quad \text{or} \quad \sin\phi = \frac{r}{l}\sin\theta$$

and $$\cos\phi = (1-\sin^2\phi)^{\frac{1}{2}} = \left(1 - \frac{r^2}{l^2}\sin^2\theta\right)^{\frac{1}{2}}$$

$$= 1 - \frac{r^2}{2l^2}\sin^2\theta, \quad \text{approximately,}$$

expanding by the binomial theorem and neglecting all terms after the second. This approximation gives sufficient accuracy for the usual values of r/l which occur in practice.

Therefore $$x = l + r - l\left(1 - \frac{r^2}{2l^2}\sin^2\theta\right) - r\cos\theta$$

$$= r(1-\cos\theta) + \frac{r^2}{2l}\sin^2\theta.$$

Velocity $$v = \frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt}$$

$$= \left\{r\sin\theta + \frac{r^2}{2l}2\sin\theta\cos\theta\right\}\frac{d\theta}{dt}$$

$$= \omega r\left\{\sin\theta + \frac{r}{2l}\sin 2\theta\right\},$$

where $$\omega = \frac{d\theta}{dt}.$$

$$\text{Acceleration } f=\frac{d^2x}{dt^2}=\frac{dv}{dt}=\frac{dv}{d\theta}\,\frac{d\theta}{dt}$$

$$=\omega r\left\{\cos\theta+\frac{r}{2l}2\cos 2\theta\right\}\frac{d\theta}{dt}$$

$$=\omega^2 r\left\{\cos\theta+\frac{r}{l}\cos 2\theta\right\}.$$

Expressions for the exact acceleration are given in Ex. 22 and Ex. 23, p. 45.

Since the acceleration is the rate of change of the velocity, the former is zero when the latter is a maximum, for the slope of the velocity-time curve is then zero. The student is asked to plot a velocity-time curve in Ex. 18, p. 44, and an acceleration-time curve in Ex. 19, p. 44.

The method of determining the velocity of the slider by a velocity diagram was explained in Art. 17, p. 19, and the acceleration diagram is dealt with in Art. 32, p. 50.

Exercises III

Take $g=32{\cdot}2$ ft./sec.^{2}

1. If a body moves along a straight line according to the law $x=t^3-2t^2+7$, using foot and second units, find the velocity and the acceleration when $t=5$ sec.

2. Given that the acceleration of a body moving along a straight line is 0·8 ft./sec.^{2}, that the velocity is 15 ft./sec. when $t=9$ sec., and that the displacement is 14 ft. when $t=0$, find the velocity and displacement when $t=16$ sec.

3. The equation $x=\sin 3t-4\cos 3t$ gives displacement x in feet at time t in seconds. Find the lowest positive value of the time when the velocity is zero and then find the value of the acceleration at that time.

4. Two trains whose lengths are 106 and 120 feet, moving in opposite directions along parallel lines, meet when their velocities are 30 and 45 miles per hour respectively. At this instant the former has an acceleration of 2 feet per sec. per sec. and the latter of 1 foot per sec. per sec. Prove that they will pass each other in two seconds. [C.U.]

5. A particle has an initial velocity $u=50$ ft./sec. along an axis OX and an acceleration $f=4$ ft./sec.^{2} in a direction inclined at 30° to OX (Fig. 47). Find the direction of motion

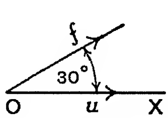

FIG. 47.

and the distance of the particle from O when 10 seconds has elapsed.

6. Two motor cyclists A and B are travelling towards towns C and D along straight roads OC and OD, the angle COD being 60°. At noon, A passes O at 10 miles per hour and has a constant acceleration of 4 miles per hour per minute, and B passes O at noon (just escaping a collision) at 20 miles per hour, and has a constant acceleration of 2 miles per hour per minute. Find the time when their distances from O are equal, and show (1) that the velocity of A relative to B is then 45·8 . . . miles per hour, and (2) that the distance between A and B is then increasing at the rate of 45 miles per hour. [C.U.]

7. A tramcar, starting from rest, is uniformly accelerated for 16 seconds to a speed of 30 miles per hour which is maintained for a distance of 280 yards, then it is uniformly retarded until it is brought to rest at a distance of 500 yards from the starting-point. Find the values of the acceleration, the retardation, and the total time for the journey.

8. A train starting from rest has an acceleration f feet per second per second, where $f = \frac{1}{1000}(t-60)^2$, and t is the time in seconds. Find, graphically or analytically, the velocity attained and the distance passed over in the first minute. [C.U.]

9. The acceleration of a vehicle, starting from rest, is given by $f = 1{\cdot}65 - 0{\cdot}0052s$ in foot and second units. Find the velocity when $s = 125$ feet and when $s = 250$ feet.

10. A train passes three consecutive mile posts A, B, and C. The time taken to travel from A to B is 80 seconds and from B to C is 56 seconds. Assuming that the acceleration is uniform, find the speeds at A and C in miles per hour and the acceleration in feet per second per second.

11. If an aeroplane travels a distance r in a steady wind along a course making an acute angle θ with the direction from which the wind blows, and returns to its starting-point, the speed of the plane in still air being u miles per hour, and the wind-velocity being v miles per hour, show that it must be *steered* in directions making equal angles a on either side of its course, where $u \sin a = v \sin \theta$.

Show also that if t_1, t_2 are the times of going and returning respectively,

$$\text{(i)}\quad r(t_1 + t_2) = 2ut_1t_2 \cos a;$$
$$\text{(ii)}\quad r(t_1 - t_2) = 2vt_1t_2 \cos \theta;$$
$$\text{(iii)}\quad r^2 = t_1t_2(u^2 - v^2).$$

[U.L.]

12. A train moving with constant acceleration is found to cover two consecutive distances of 200 yards in 10 and 9 seconds respectively. Determine the acceleration of the train, and find its velocity at the end of the first of these distances. [C.U.]

13. A shell is fired from a gun with an initial velocity of 1000 feet per second and at an angle of 30° to the horizontal. Neglecting air resistance, find the time of flight, the range, and the greatest height of the shell, assuming it strikes the ground at a point which is on the same level as the gun.

14. Find the time of flight in the preceding exercise if the point of impact is 250 feet lower than the level of the gun. Also, find the distance of the gun from the vertical line through the point of impact.

15. A particle is projected with a velocity of 120 feet per second and just clears a vertical wall 25 feet higher than the point of projection and at a distance of 300 feet from that point. Find the two possible angles of projection and the corresponding distances, beyond the wall, where the particle hits the ground at the same level as the point of projection. [C.U.]

16. A projectile has an initial velocity u at an angle of elevation α. Neglecting air resistance, show that the horizontal range R is a maximum when $\alpha = 45°$, that $R_{max} = u^2/g$, and that the corresponding maximum height is $\frac{1}{4}R_{max}$.

Find the time of flight of a ball which is thrown, at an angle of elevation of 45°, a horizontal distance of 50 yards.

17. If a projectile fired from the ground level just clears an obstacle of height b at a horizontal distance a, and reaches the ground again at distance c from its starting-point, prove that the angle of projection is $\tan^{-1} \dfrac{bc}{a(c-a)}$.

From this formula, or otherwise, verify the following construction: If A is the starting-point, B the top of the obstacle, and C the farthest point, draw CB and produce it to cut the vertical through A at D, draw DE horizontal to cut the vertical through B at E: then AE is the direction of projection [U.L.]

18. The displacement of a piston from the beginning of its stroke is given approximately by

$$x = 0{\cdot}230(1 - \cos 50\pi t) + 0{\cdot}026 \sin^2 50\pi t,$$

where x is in feet and the time t is in seconds.

Show that this equation may be written as

$$x = 0{\cdot}243 - 0{\cdot}230 \cos 50\pi t - 0{\cdot}013 \cos 100\pi t,$$

then, by differentiating either form, obtain an equation for the velocity of the piston.

Find the velocity when t has the values 0, $\frac{1}{200}$, $\frac{1}{100}$, $\frac{3}{200}$, and $\frac{1}{50}$ second, then, plotting these results, sketch the velocity-time graph.

19. If the velocity of the piston in the preceding exercise is

$$\frac{dx}{dt} = 11{\cdot}50\pi \sin 50\pi t + 1{\cdot}3\pi \sin 100\pi t,$$

find the acceleration when t has the values 0, $\frac{1}{200}$, $\frac{1}{100}$, $\frac{3}{200}$, and $\frac{1}{50}$ second, and plot the acceleration-time curve.

20. Given that the velocity of a piston is approximately

$$v = \omega r\left\{\sin\theta + \frac{r}{2l}\sin 2\theta\right\},$$

where ω is the uniform angular velocity of the crank, θ is the crank angle, and r and l are respectively the lengths of the crank and connecting-rod, find to the nearest degree the first value of θ when v is a maximum, taking $l = 5r$.

21. In the slider-crank mechanism, Fig. 46, p. 41, $l\sin\phi = r\sin\theta$. Assuming that $\frac{d\theta}{dt}$ or ω, the angular velocity of the crank, is uniform, show by differentiation that the angular velocity and the angular acceleration of the connecting-rod are respectively

$$\frac{d\phi}{dt} = \frac{\omega r\cos\theta}{(l^2 - r^2\sin^2\theta)^{\frac{1}{2}}},$$

and

$$\frac{d^2\phi}{dt^2} = \frac{\omega^2 r(r^2 - l^2)\sin\theta}{(l^2 - r^2\sin^2\theta)^{\frac{3}{2}}}.$$

22. Given that the displacement of a piston from the beginning of its stroke is

$$x = l + r - l\left\{1 - \frac{1}{n^2}\sin^2\theta\right\}^{\frac{1}{2}} - r\cos\theta,$$

where $n = l/r$, show that an exact expression for the acceleration is

$$\frac{d^2x}{dt^2} = \omega^2 r\left\{\frac{\sin^2 2\theta}{4(n^2 - \sin^2\theta)^{\frac{3}{2}}} + \frac{\cos 2\theta}{(n^2 - \sin^2\theta)^{\frac{1}{2}}} + \cos\theta\right\},$$

where the angular velocity $\omega = \frac{d\theta}{dt}$ is uniform.

Then show that when the acceleration is zero

$$\sin^6\theta - n^2\sin^4\theta - n^4\sin^2\theta + n^4 = 0.*$$

23. For a direct acting engine CP is the crank, PD the connecting-rod, and the angles PCD and PDC are θ and ϕ respectively. If the crank rotates with a uniform angular velocity ω, prove that the acceleration of D is given by the expression

$$\frac{\omega^2 r}{k}\left\{\cos\phi + k\cos\theta - \frac{1-k^2}{\cos^3\phi}\right\},$$

where k is the ratio of CP to PD and r is the length CP. Show

* See D. A. Low's *Applied Mechanics*, p. 304, for roots of this equation for various values of n.

that a close approximation to the acceleration of D is given by

$$\omega^2 r\left\{\cos\theta+\left(k+\frac{k^3}{4}\right)\cos 2\theta-\frac{k^3}{4}\cos 4\theta\right\}. \qquad \text{[C.U.]}$$

24. Fig. 48 shows a mechanism used for giving an intermittent motion to the film in a projection apparatus. A pin C, fixed to a wheel, rotating about an axis A with uniform angular velocity ω, engages with a slotted piece which rotates about an axis B. The slots are at right angles to one another and the ratio of AB to AC is $\sqrt{2}$. Show that during the motion of the slotted piece its angular velocity is given by $\dfrac{\sqrt{2}\cos\theta-1}{3-2\sqrt{2}\cos\theta}\omega$, where θ is the angle CAB.

Find similar expressions for the velocity of sliding of the pin in a slot, and the angular acceleration of the slotted piece. [C.U.]

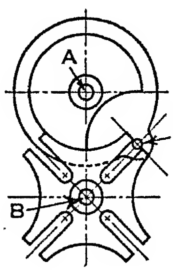

FIG. 48.

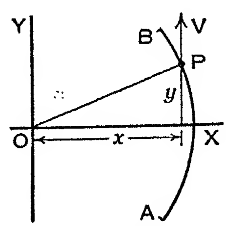

FIG. 49.

25. During the operation of a cam mechanism a point P moves along a short circular arc AB (Fig. 49) whose equation is $x^2+y^2=r^2$. It is arranged that the velocity in the direction parallel to the axis OY is constant and equal to V. Show that acceleration in the direction parallel to the axis OX is $-V^2r^2/x^3$.

CHAPTER IV

ACCELERATION DIAGRAMS

28. Radial Acceleration of a Point Moving in a Circular Path.—Let P and Q (Fig. 50) be two positions near together of a point moving with a velocity v of constant magnitude, in a circular path with centre C and of radius r, and let δt be the time taken to move from P to Q.

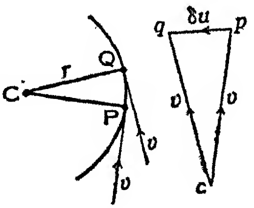

FIG. 50.

Since at any instant the direction of the velocity is tangential to the circle, by drawing the velocity triangle cpq it can be seen that in passing from P to Q the moving point has its velocity changed by an amount pq, due to the change in the direction from cp to cq. Let this change of velocity be denoted by δu.

If Q is very near to P, the arc PQ may be regarded as a straight line and then CPQ and cpq are similar triangles with cp, cq, and pq perpendicular to CP, CQ, and PQ, respectively.

Therefore $\frac{pq}{cp}=\frac{\mathrm{PQ}}{\mathrm{CP}}$ or $\frac{\delta u}{v}=\frac{\mathrm{PQ}}{r}$, approximately, but

$\mathrm{PQ}=v\delta t$, therefore $\frac{\delta u}{v}=\frac{v\delta t}{r}$ or $\frac{\delta u}{\delta t}=\frac{v^2}{r}$, approximately.

The nearer Q is to P the more nearly exact does this approximation become. If δt is made to approach zero, Q becomes indefinitely near to P and $\frac{\delta u}{\delta t}$ approaches the value $\frac{du}{dt}$ which will be the acceleration of the point P. This acceleration acts along the radius towards C the centre

of the circle and is called the *centripetal acceleration* of the point P.

Therefore the centripetal acceleration is

$$\frac{du}{dt}=\frac{v^2}{r}=\omega^2 r,$$

where $\omega=v/r$ is the angular velocity of the radius CP.

29. Radial Acceleration of a Point Moving in any Curved Path.—The arguments of the preceding Art. may be applied to a point P moving with a velocity v of constant magnitude along any curved path, by taking the point C (Fig. 50) as the centre of curvature of the curve at the point P. Then CP$=r$ is the radius of curvature of the curve at the point P. The length CQ is not equal to r, but approaches this value as Q approaches P, therefore CPQ and *cpq* may be regarded as similar triangles and the proof continued as before.

The centripetal acceleration of the point P is v^2/r, where v is its velocity along the curve and r is the radius of curvature of the curve at the point P. The formula is also true when the magnitude of the velocity is varying.

30. Resultant Acceleration of a Point on a Body Rotating with Varying Velocity.—Let A and B (Fig. 51) be points on a rigid body moving with plane motion so that the line AB rotates in the plane of the paper with varying angular velocity ω about the point A, which is fixed, and let the length AB be r.

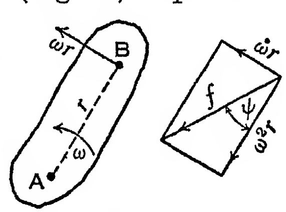

FIG. 51.

At any instant the velocity of the point B is ωr along the tangent to the path of B—that is, perpendicular to AB.

The angular acceleration of AB is $\frac{d\omega}{dt}$ which will be written as $\dot{\omega}$, and the acceleration of B perpendicular to AB is $\dot{\omega}r$. The sense of the acceleration $\dot{\omega}r$ is the same as the sense of the velocity ωr if this velocity is increasing.

The acceleration of B along BA is $\omega^2 r$. Drawing the acceleration diagram for the point B and denoting the resultant acceleration of B by the symbol f, then it can be seen that

$$f=\sqrt{\dot{\omega}^2 r^2+\omega^4 r^2}=r\sqrt{\dot{\omega}^2+\omega^4}$$

and this resultant acts in a direction inclined to BA at an angle ψ, where $\psi=\tan^{-1}(\dot{\omega}/\omega^2)$.

A velocity is always in the direction of motion, but a resultant acceleration cannot be continually in the direction of motion unless the motion is in a straight line.

31. **Accelerations of Points on a Rigid Body—Acceleration Image.**—Suppose the accelerations of two points A and B (Fig. 52) on a rigid body having plane motion are known, and let them be f_a and f_b, respectively, the directions being as shown. The points A and B are assumed to be in the plane of the paper, which is a plane of motion.

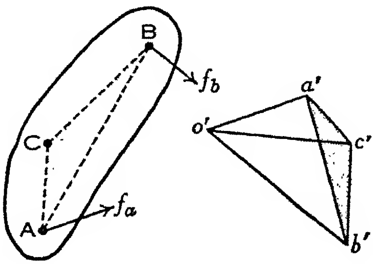

FIG. 52.

In constructing a velocity diagram a pole o was used, and a vector oa represented the velocity of a point A. In an acceleration diagram the pole will be designated o', and a vector $o'a'$ will be used to represent the acceleration of a point A.

Therefore, from a convenient pole o' mark off $o'a'$ and $o'b'$ to represent the given accelerations in direction and magnitude. Join $a'b'$, then $a'b'$ represents the acceleration of B relative to A, and $b'a'$ represents the acceleration of A relative to B.

The vector equation is

$$o'a'=o'b'+b'a'.$$

In words—

Acceleration of A = *Acceleration of* B
+ *Acceleration of* A *relative to* B.

To find the acceleration of any other point C on the body and in the plane of the paper, join AC and BC and draw the triangle $a'b'c'$ similar to the triangle ABC. Join $o'c'$, then $o'c'$ represents the acceleration of C. The triangle $a'b'c'$ is called the *acceleration image* of the triangle ABC, or the line $a'b'$ is the acceleration image of the line AB.

In practice the work is not so easy as in the foregoing description, since it is unusual for both f_a and f_b to be known completely. Suppose f_b is known in magnitude and direction, but f_a is known only in direction and it is required to find its magnitude. The length and direction of $o'b'$ is known as before and the direction $o'a'$ is known, but the length of $o'a'$ is unknown. The point a' is found by considering the acceleration of A relative to B. An example is given in the next Art.

32. Accelerations in the Slider-Crank Mechanism.—The crank CB (Fig. 53) rotates with constant angular velocity ω and it is required to find the acceleration of the slider A and the angular acceleration of the connecting-rod AB when the mechanism is in the given position. (The analytical determination of the acceleration of the slider has been given in Art. 27.)

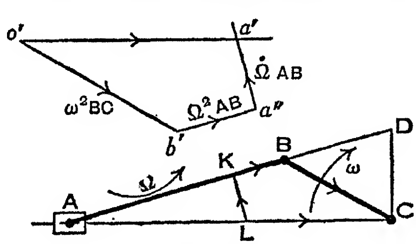

FIG. 53.

Let Ω be the angular velocity of the connecting-rod AB. Produce AB to intersect at D a line drawn from C perpendicular to AC, then, as shown in Art. 17, $\Omega=\omega\frac{BD}{AB}$.

To find the acceleration of the slider A, the acceleration diagram is drawn with the help of the vector equation—

Acceleration of A = Acceleration of B
+ Acceleration of A relative to B.

From a convenient pole o' draw $o'b'$ parallel to BC to represent, to a suitable scale, ω^2BC the acceleration of B.

The acceleration of A relative to B consists of two components, one along AB towards B and equal to Ω^2AB and another perpendicular to AB and equal to $\dot{\Omega}$AB. Now Ω^2AB can be calculated, therefore draw $b'a''$ parallel to AB to represent Ω^2AB, then draw $a''a'$ perpendicular to AB. Since $\dot{\Omega}$ is unknown, the magnitude of $\dot{\Omega}$AB cannot be calculated and therefore the length of $a''a'$ is as yet unknown. However, it is known that the acceleration of A is along AC, therefore the closing line $o'a'$ of the acceleration diagram can be drawn parallel to AC to intersect $a''a'$ at a'.

Therefore the acceleration of A is $o'a'$ measured with the acceleration scale. The angular acceleration of AB is $\dot{\Omega}=\dfrac{a''a'}{\text{AB}}$, where $a''a'$ is measured with the acceleration scale. The arrowhead on the vector $a''a'$ shows the direction of this component of the acceleration of A relative to B, and therefore it follows that the direction of $\dot{\Omega}$ is clockwise. Since the angular velocity of AB is anti-clockwise, the clockwise acceleration indicates a retardation.

The acceleration diagram may also be drawn to a particular scale on the mechanism diagram ABC.

Since $\Omega=\omega\dfrac{\text{BD}}{\text{AB}}$, therefore $\Omega^2\text{AB}=\omega^2\dfrac{(\text{BD})^2}{\text{AB}}$. Along BA mark off a point K so that $\text{KB}=\dfrac{(\text{BD})^2}{\text{AB}}$ and draw KL at right angles to AB to intersect AC at L. Then ω^2LC is the acceleration of A, or LC represents the acceleration of A to the same scale that BC represents the acceleration of B.

The proof, put briefly, is as follows. The figures CBKL and $o'b'a''a'$ are similar since each side in one is parallel to a side in the other, taken in order, and since BC is proportional to ω^2BC or to $o'b'$ and KB is proportional to ω^2KB or Ω^2AB or proportional to $b'a''$. Therefore the acceleration of A is ω^2LC. Also, $\omega^2\text{LK}=\dot{\Omega}\text{AB}$ and therefore $\dot{\Omega}=\omega^2\dfrac{\text{LK}}{\text{AB}}$. Constructions for drawing the line KL are given in the next two Arts.

33. Klein's Construction.—The slider-crank mechanism is shown again in Fig. 54. Produce AB to intersect at D a line drawn from C perpendicular to AC. Draw a circle with AB as diameter. With centre B and radius BD draw another circle cutting the first circle at M and N. Join MN, intersecting AB at K and AC at L.

Since MN is a chord common to both circles and since both circles have their centres in AB, therefore MN is perpendicular to AB. Join AM and BM. The right-angled triangles KBM and MBA are similar, therefore $\frac{KB}{BM}=\frac{BM}{AB}$ or $KB=\frac{(BM)^2}{AB}$. But BM=BD, therefore KB $=\frac{(BD)^2}{AB}$. Therefore, as shown in the preceding Art., the acceleration of the slider A is ω^2LC. The student's attention is drawn to Ex. 6, p. 61, where the angular velocity of the crank is not uniform.

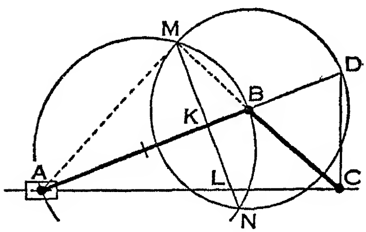

FIG. 54.

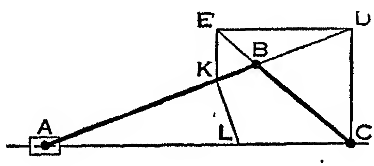

FIG. 55.

34. Mohr's Construction.*—As in Klein's construction, produce AB to intersect at D a line drawn from C perpendicular to AC (Fig. 55). Draw DE parallel to CA to intersect CB produced at E. Draw EK parallel to DC, intersecting AB at K, then draw KL perpendicular to AB, intersecting AC at L.

The triangles EBK and CBD are similar, therefore $\frac{KB}{BD}=\frac{EB}{BC}$. Also the triangles EBD and CBA are similar, therefore $\frac{EB}{BC}=\frac{BD}{AB}$.

* Also attributed to *Rittershaus* and others.

Therefore $\frac{KB}{BD}=\frac{BD}{AB}$ or $KB=\frac{(BD)^2}{AB}$, therefore K is the required point in AB and, since KL is perpendicular to AB, L is the required point in AC. Therefore the acceleration of the slider A is ω^2LC.

35. Centre of Zero Acceleration.—Let the accelerations of points A and B on a rigid body having plane motion (Fig. 56) be f_a and f_b in the directions Aa and Bb, respectively, and let $o'a'b'$ be the acceleration diagram. It is assumed that the points A and B are in the plane of the paper, which is a plane of motion. The acceleration of any point C on the body, and in the plane of the paper, is represented on the acceleration diagram by $o'c'$ if the triangle $a'b'c'$ is the image of the triangle ABC (Art. 31).

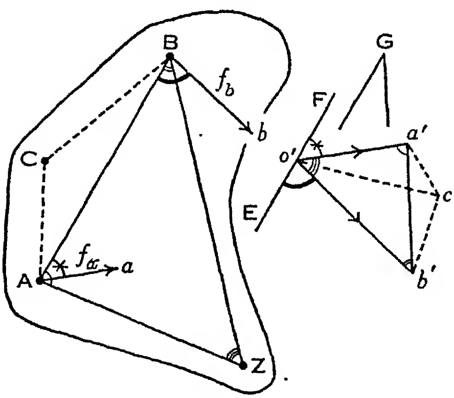

FIG. 56.

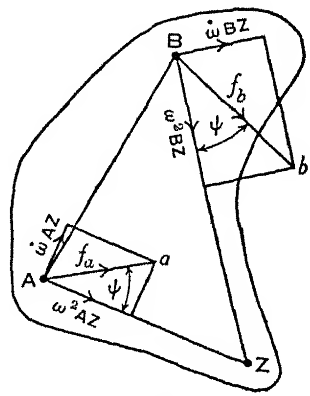

FIG. 57.

If the point c' coincided with the pole o', then the acceleration of C would be zero. Therefore draw the triangle ABZ so that the triangle $a'b'o'$ is its image, then the pole o' is the image of the point Z. The point Z has no acceleration and it is called the *centre of zero acceleration* or the *acceleration centre.*

It will now be shown that the angle ZAa is equal to the angle ZBb. Draw EF through o' and parallel to AB. Produce EF and $b'a'$ to intersect at G. (In the figure the lines intersecting at G have been drawn parallel to EF and

$b'a'$, respectively.) The angles which are known to be equal are indicated by similar marks.

$$\text{Angle } ZAa = \text{angle } ZAB - \text{angle } aAB$$
$$= \text{angle } o'a'b' - \text{angle } Fo'a' = \text{angle at G.}$$
$$\text{Angle } ZBb = \text{angle } ABb - \text{angle } ABZ$$
$$= \text{angle } Eo'b' - \text{angle } o'b'a' = \text{angle at G.}$$

Therefore $\quad$ angle ZAa = angle ZBb.

The position of the point Z may be arrived at in another way. Suppose the body has an angular velocity ω and an angular acceleration $\dot{\omega}$. Draw AZ (Fig. 57) so that the angle $ZAa = \psi = \tan^{-1}\frac{\dot{\omega}}{\omega^2}$, then the length of AZ may be determined as follows:—

Acceleration of A = Acceleration of Z + Acceleration of A relative to Z.

Therefore f_a = Acceleration of Z + $AZ\sqrt{\omega^4+\dot{\omega}^2}$.

Make the length AZ such that $f_a = AZ\sqrt{\omega^4+\dot{\omega}^2}$, then the acceleration of the point Z will be zero, and Z will be the centre of zero acceleration.

Now join BZ. Since the acceleration of Z is zero, the acceleration of B is $f_b = BZ\sqrt{\omega^4+\dot{\omega}^2}$ and the angle $ZBb = \psi = \tan^{-1}\frac{\dot{\omega}}{\omega^2}$. The acceleration of any other point may be found in a similar way.

The point Z may be obtained by drawing AZ and BZ inclined at the angle ψ to Aa and Bb, respectively, and intersecting at Z. If, as shown, f_b is greater than f_a, then ZB is greater than ZA and therefore the angles ψ are measured in clockwise directions from the lines Bb and Aa, for it can be seen that ZB would be less than ZA if the angles ψ were measured in anticlockwise directions. If only one acceleration is given, or if the directions of two accelerations are given without their magnitudes, then the direction of $\dot{\omega}$ must be known in order to determine the direction of angle ψ. The angle ψ is measured clockwise

or anticlockwise from the direction of a given acceleration according as $\dot{\omega}$ is clockwise or anticlockwise.

The point Z may be inside or outside a body, but in the latter case it must still be regarded as rigidly attached to the body.

36. Acceleration of a Point Moving along a Rotating Straight Line.—Let P be a point moving along a straight line OP which is rotating about O (Fig. 58). It is required to find the acceleration of the point P along and perpendicular to OP.

Suppose that at a given instant OP$=r$, OP is inclined at an angle θ to a fixed line OX, the angular velocity of OP is $\frac{d\theta}{dt}$ or ω, and the velocity of P along OP is $\frac{dr}{dt}$ or u. Also, suppose that during a short time δt, θ increases to

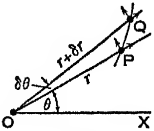

Fig. 58.

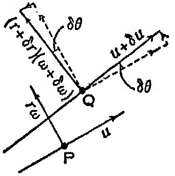

Fig. 59.

$\theta+\delta\theta$, P moves to Q, so that r increases to $r+\delta r$, ω becomes $\omega+\delta\omega$, and u becomes $u+\delta u$.

At P the velocity perpendicular to OP is $r\omega$, and at Q the velocity perpendicular to OQ is $(r+\delta r)(\omega+\delta\omega)$.

If the velocities at Q are resolved in directions parallel and perpendicular to OP, as indicated in the enlarged view (Fig. 59), then the changes in the velocities of P in these directions, and consequently the accelerations of P, may be determined.

Parallel to OP—

$$\frac{\text{Change of vel.}}{\text{Time}} = \frac{(u+\delta u)\cos\delta\theta - (r+\delta r)(\omega+\delta\omega)\sin\delta\theta - u}{\delta t}$$

Let Q approach P, that is let δt approach zero, then $\delta\theta$ approaches zero, cos $\delta\theta$ approaches 1, sin $\delta\theta$ approaches $\delta\theta$, the products $r\delta\omega\frac{\delta\theta}{\delta t}$, $\omega\delta r\frac{\delta\theta}{\delta t}$, $\delta r\,\delta\omega\frac{\delta\theta}{\delta t}$ ultimately vanish, $\frac{\delta u}{\delta t}$ is written $\frac{du}{dt}$ and $\frac{\delta\theta}{\delta t}$ is written $\frac{d\theta}{dt}$ or ω, and the expression becomes the acceleration of P along OP, therefore

$$\text{Acceleration of P along OP} = \frac{du}{dt} - \omega^2 r \quad . \quad (1).$$

Perpendicular to OP—

$$\frac{\text{Change of vel.}}{\text{Time}} = \frac{(u+\delta u)\sin\delta\theta + (r+\delta r)(\omega+\delta\omega)\cos\delta\theta - r\omega}{\delta t}.$$

Let δt approach zero, then the expression becomes the acceleration of P perpendicular to OP. Therefore, in a similar manner to the previous simplification,

Acceleration of P perpendicular to OP

$$= u\frac{d\theta}{dt} + \frac{dr}{dt}\omega + r\frac{d\omega}{dt} = 2u\omega + r\frac{d\omega}{dt} \qquad (2),$$

since $$\frac{d\theta}{dt} = \omega \quad \text{and} \quad \frac{dr}{dt} = u.$$

Expression (1) might have been written down straightaway, for it consists of $\frac{du}{dt}$, the rate of change of the velocity of P along OP, and the centripetal acceleration $\omega^2 r$ due to the rotation of OP. In expression (2) the term $r\frac{d\omega}{dt}$ was also expected; the term $2u\omega$ may astonish the student, but it is accounted for by the fact that OP is rotating whilst changing in length. The acceleration $2u\omega$ is sometimes called the *compound supplementary acceleration* or the *Coriolis acceleration.*

The importance of these results is illustrated by the example worked out in the next Art., and another method of obtaining them is suggested in Ex. 17, p. 65.

37. Shaping Machine Mechanism.—A mechanism for a shaping machine is shown diagrammatically in Fig. 60. A crank CB turning about a fixed centre C drives a slider through a pin B. The slider slides on a link AE turning about a fixed centre A and causes AE to oscillate and, through a link EF, drive a slider at F to and fro along the line HF. The various dimensions in inches are: AC = 5, CH = $4\frac{1}{2}$, CB = $2\frac{1}{2}$, AE = 11, and EF = 8. The angle CHF = 90° and the angle HCB = 30°. The crank CB is turning uniformly at 30 r.p.m., and it is required to find the velocity and acceleration of the slider at F when the mechanism has the given configuration.

Let the angular velocities of CB, AE, and EF, in radians per second, be denoted by ω, Ω_1, and Ω_2, respectively. Also let D be the point on the link AE immediately under the pin B. For an instant B and D are coincident, but B slides away from D as the crank rotates. Let the velocity of B relative to D be u.

The angular velocity of CB is $\omega = 30 \times \frac{2\pi}{60} = \pi$ rad./sec.

The velocity of B, perpendicular to CB, is $\pi \times \frac{2 \cdot 5}{12} = 0 \cdot 654$ ft./sec.

The velocity diagram may now be constructed. The scale used in Fig. 60 is 1 inch = 0·5 ft./sec., but the student should work the problem on drawing-paper, using a scale of say 1 inch = 0·1 ft./sec. Starting with a convenient pole o, draw ob perpendicular to CB to represent to scale the velocity of B. Since the point D moves in the direction perpendicular to AE and the point B moves along AE, draw od perpendicular to AE and draw bd parallel to AE, meeting od at d, then od represents the velocity of D and db represents u, the velocity of B relative to D.

Produce od to e, making $\frac{oe}{od} = \frac{\text{AE}}{\text{AD}}$, then oe represents the velocity of E. The construction is shown by the dotted lines; os makes any convenient angle with od, os = AE and or = AD, rd is joined and se is drawn parallel to rd to meet oe at e.

Since the point F moves along HF and its motion relative to E is perpendicular to FE, draw *of* parallel to HF and draw *ef* perpendicular to EF, meeting *of* at *f*, then *ef* represents the velocity of F relative to E and *of* represents the velocity of F. Measuring *of* with the velocity scale it is found that the velocity of F is 0·89

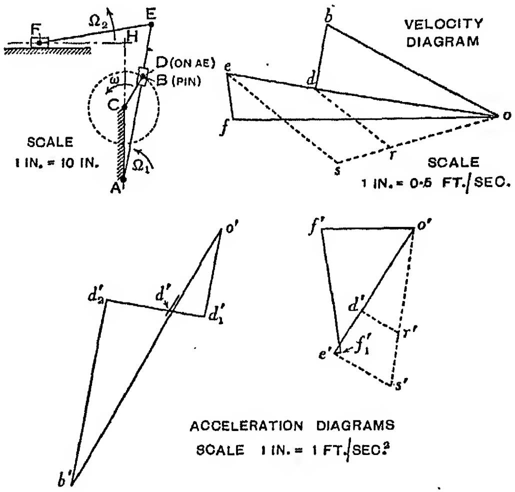

Fig. 60.

ft./sec. Also the velocity of B relative to D is *db* or $u = 0{\cdot}225$ ft./sec.

Two acceleration diagrams are shown, but this is only for clearness; they should be combined and drawn as one diagram. The acceleration of the point B may be expressed in two ways: as a point on the crank CB it is ω^2CB acting towards C, and as a point moving along the link AE it is $\dot{u} - \Omega_1^2$AD along AE in the sense A to E and $\dot{\Omega}_1$AD $+ 2u\Omega_1$ perpendicular to AE in the sense in which the point D is moving, provided in each case the numerical

result is positive. It will be found later that $\dot{u}$ is negative. It should be noted that $-\Omega_1^2$AD and $\dot{\Omega}_1$AD are the accelerations of the point D, and although Ω_1 is easily calculated, $\dot{\Omega}_1$ is more troublesome to find. Once the resultant acceleration of D is obtained, the accelerations of E and F may be found without difficulty.

The acceleration scale is 1 inch = 1 ft./sec.^{2}, but the student should use a much larger scale, say 1 inch = 0·2 ft./sec.^{2}.

The acceleration of B is $\omega^2\text{CB} = \pi^2 \times \frac{2\cdot5}{12} = 2\cdot06$ ft./sec.^{2}, therefore, from a convenient pole o' draw $o'b'$ parallel to BC and equal to 2·06 inches.

Now $\Omega_1 = \frac{od}{\text{AD}}$. By measurement $od = \frac{1\cdot23}{2}$ ft./sec., and AD = 7·3 inches, therefore $\Omega_1 = \frac{1\cdot23}{2} \times \frac{12}{7\cdot3} = 1\cdot01$ rad./sec.

Also $\Omega_1^2\text{AD} = 1\cdot01^2 \times \frac{7\cdot3}{12} = 0\cdot62$ ft./sec.^{2} and the direction is towards A, therefore draw $o'd_1'$ parallel to DA and equal to 0·62 inch.

Next draw $d_1'd_2'$ perpendicular to AD, then this is the direction of the acceleration $\dot{\Omega}_1$AD, the magnitude and sense of which are unknown, and it is the direction of the acceleration $2u\Omega_1$ the magnitude of which may be calculated. The sense of $2u\Omega_1$ is from d_1' to d_2' because, as seen from the velocity diagram, $u = db$ is positive or in the sense A to E.

The acceleration of B relative to D along AE is $\dot{u}$, therefore draw $b'd_2'$ parallel to AE, intersecting $d_1'd_2'$ at d_2', then $d_2'b'$ represents $\dot{u}$, which is evidently a retardation since the sense is opposite to the sense of the velocity u.

As already found, $u = 0\cdot225$ ft./sec. and $\Omega_1 = 1\cdot01$ rad./sec., therefore $2u\Omega_1 = 2 \times 0\cdot225 \times 1\cdot01 = 0\cdot455$ ft./sec.^{2}. From d_2' mark off $d_2'd' = 0\cdot455$ inch along $d_2'd_1'$, then $d'd_2'$ represents $2u\Omega_1$ and, by subtraction, $d_1'd'$ represents $\dot{\Omega}_1$AD.

Join $o'd'$, then this is the total acceleration of the point D. To prevent confusion the line $o'd'$ has been drawn in the

other diagram on the right, but as previously mentioned the two diagrams should be drawn as one.

Produce $o'd'$ to e' making $\frac{o'e'}{o'd'}=\frac{\text{AE}}{\text{AD}}$. The construction is as shown by the dotted lines where $o'r'=\text{AD}$, $o's'=\text{AE}$, $r'd'$ is joined and $s'e'$ is drawn parallel to $r'd'$ to meet $o'd'$ produced at e'. The acceleration of E is represented by $o'e'$.

The acceleration of F may now be obtained.

Acceleration of F = Acceleration of E
+ Acceleration of F relative to E.

The acceleration of F is along the line HF, therefore draw $o'f'$ parallel to HF, the length of $o'f'$ being unknown. The acceleration of F relative to E is $\Omega_2^2\text{FE}$ along FE towards E and $\dot{\Omega}_2\text{FE}$ perpendicular to FE, where Ω_2 and $\dot{\Omega}_2$ are respectively the angular velocity and angular acceleration of FE.

$$\text{Now } \Omega_2=\frac{fe}{\text{FE}} \text{ and } \Omega_2^2\text{FE}=\frac{(fe)^2}{\text{FE}}=0{\cdot}16^2\times\frac{12}{8}=0{\cdot}038 \text{ ft./sec.}^2,$$

the value of fe being obtained from the velocity diagram. Therefore draw $e'f_1'$ parallel to FE and equal to 0·038 inch, then draw $f_1'f'$ perpendicular to FE and intersecting $o'f'$ at f'. The length $f_1'f'$ represents $\dot{\Omega}_2\text{FE}$. The acceleration of F is represented by $o'f'$, and by measurement with the acceleration scale it is found to be 0·62 ft./sec.^{2}, approximately.

Exercises IV

1. A vertical single cylinder engine running at 1800 revolutions per minute has a crank $2\frac{1}{4}$ inches long and a connecting-rod $10\frac{1}{2}$ inches long, and the line of stroke of the piston passes through the centre of the crankshaft. Draw the acceleration diagram when the crank is at an angle of 30° from the top dead centre, and state the magnitude of the acceleration of the piston in feet per second per second.

2. A crank BC of length r feet is turning about the point C at ω radians per second and is connected to a slider A by a rod AB

of length l feet, the line of stroke of A being along AC. Draw acceleration diagrams to find the acceleration of the slider, (*a*) when the angle ACB is 0°, (*b*) when the angle ACB is 180°.

3. The slider-crank mechanism as shown in Fig. 61 has the line of stroke of the slider A offset a perpendicular distance of 2 inches from the centre C. AB = 30 inches, BC = 8 inches, and BC is rotating clockwise at 200 r.p.m. Draw an acceleration diagram to find the acceleration of the slider when the angle DCB is 60°, the datum line DC being parallel to the line of stroke.

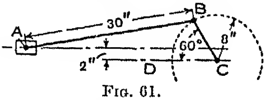

FIG. 61.

4. Referring to the preceding exercise and Fig. 61, find the acceleration of the slider A for each of the positions when AB and BC are in line, (*a*) not overlapping, (*b*) overlapping.

5. In the slider-crank mechanism ABC (Fig. 62) the line KL is obtained as explained in Art. 33 and BL is joined. From any point E in AB, E*e* is drawn parallel to AC to intersect BL at *e* and *e*C is joined. If BC rotates with a uniform angular velocity ω, show that the acceleration of the point E is $\omega^2 . eC$.

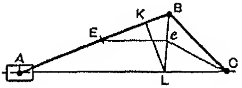

FIG. 62.

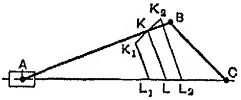

FIG. 63.

6. Let ω and $\dot{\omega}$ be the angular velocity and angular accelera- of the crank BC in the slider-crank mechanism ne KL is drawn as explained in Art. 33, then if $\dot{\omega} = 0$ the acceleration of the slider A is ω^2LC. Draw K_1K_2 through K and perpendicular to BC, making $KK_1 = KK_2 = \dfrac{\dot{\omega}BC}{\omega^2}$, then draw K_1L_1 and K_2L_2 parallel to KL to meet AC at L_1 and L_2.

Draw acceleration diagrams and show that the acceleration of A is $\omega^2 . L_1C$ when the angular acceleration $\dot{\omega}$ is clockwise and $\omega^2 . L_2C$ when $\dot{\omega}$ is anticlockwise.

7. Referring to Fig. 63 and the preceding exercise, show that the angular acceleration of the rod AB is zero for the given configuration if $\dot{\omega} = \dfrac{AK}{AB}\omega^2 \tan\theta$ and is clockwise, θ being the angle ACB.

8. When a link AB, which is 6 inches long, is in the given position (Fig. 64) its angular motion is clockwise and the velocity and acceleration of the point A and the acceleration of the point B are as indicated. Find the angular velocity and angular acceleration of the link and the velocity of the point B.

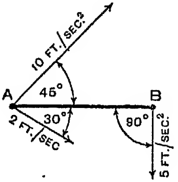

FIG. 64.

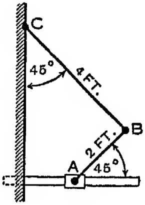

FIG. 65.

9. In the mechanism illustrated in Fig. 65 the slider A is moving to the right with a velocity v and an acceleration a, prove that the horizontal and vertical components of the acceleration of B are $\frac{a}{2}-\frac{3v^2}{8\sqrt{2}}$ and $\frac{a}{2}-\frac{v^2}{8\sqrt{2}}$ respectively. [C.U.]

10. Fig. 66 illustrates the arrangement of the crank and connecting-rods of each pair of cylinders of a 12-cylinder 60° Vee engine. Draw velocity and acceleration diagrams for the mechanism, and find the velocity and acceleration of the pistons and the angular velocity and angular acceleration of the

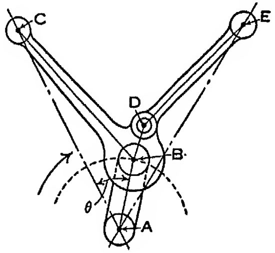

FIG. 66.

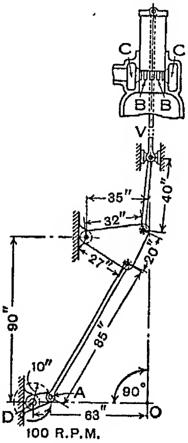

FIG. 67.

connecting-rods when the crank angle θ is 40°. Engine speed, 2400 r.p.m.

AB = 2·5 inches, BC = 6·25 inches, BD = 1·25 inches, DE = 5 inches, angle CBD = 60°. (Scales—In vector diagrams represent velocity and acceleration of B by 2·5 inches.) [C.U.]

11. Fig. 67 is a diagram of a cylindrical scavenge valve and the gear for operating it in the cylinder head of a large two-stroke Diesel engine. The valve-rod V is operated from the auxiliary crank DA of radius 10 inches which rotates at a constant speed of 100 r.p.m., and the scavenge air enters the cylinder from the receiver C through the ring of ports BB.

Draw velocity and acceleration diagrams for the mechanism for the position in which the crank pin A is above the line DO and the angle ADO is 15°, and find the velocity and the acceleration of the valve-rod V. (Scales for vector diagrams—1 inch = 2 feet per second; 1 inch = 15 feet per second per second.) [C.U.]

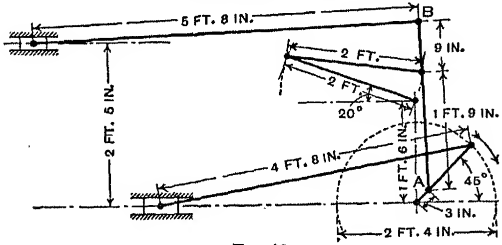

FIG. 68.

12. Draw the diagram of Marshall's valve gear as shown in Fig. 68. If the crank is 14 inches long and rotates at 100 r.p.m., draw the velocity diagram and the acceleration diagram; find the angular velocity and angular acceleration of the link AB. [U.L.]

13. Fig. 69 is a diagrammatic representation of the crank and slotted lever mechanism as applied to a shaping machine. Neglecting the obliquity of EF and assuming in consequence that the acceleration of F is the same as the acceleration of E in the direction at right angles to CD, show how to find this acceleration for [illegible] crank CP when it is [illegible]

For the position in which PCD is

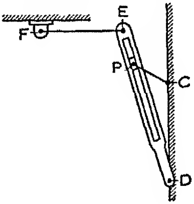

FIG. 69.

a right angle prove that the acceleration of F is

$$\frac{n^6+1}{(n^2+1)^{\frac{3}{2}}}\omega^2 . \mathrm{DE},$$

where n is the ratio of $\frac{\mathrm{DC}}{\mathrm{CP}}$ and ω is the angular velocity of the crank. [C.U.]

14. Draw velocity and acceleration diagrams for the quick-return mechanism (Fig. 70) in the position shown, when the crank BC is rotating clockwise at a uniform speed of 60 revolutions per minute.

The slotted lever AD and the crank BC turn about fixed centres A and B respectively. BC = 4 inches. DE = 6 inches. The line of stroke of E is at right angles to AB and at a perpendicular distance of 12 inches from A.

(Scales—1 inch = 5 inches per second; 1 inch = 25 inches per second per second.) [C.U.]

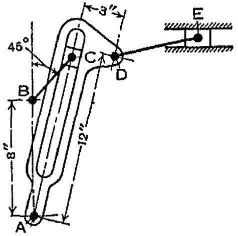

Fig. 70.

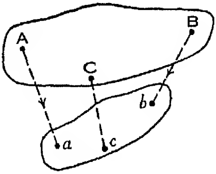

Fig. 71.

15. ABC (Fig. 71) is a link of a mechanism moving in a plane. Aa and Bb represent in magnitude, direction, and sense the accelerations of the points A and B respectively. On ab an image of the link ABC is constructed. Prove that if the triangle acb is similar to the triangle ACB, then the acceleration of C is completely represented by the vector Cc.

Prove, also, that a similar construction will give the velocity of any point of the link. [C.U.]

16. In a four-bar chain ABCD, in which AD is the fixed link, AB and DC intersect at I and CB and DA intersect at K. (See Fig. 72 which was not in the original examination question.) AN and AM are drawn parallel to DC and BC respectively and cut these lines in N and M. F is the point of intersection of KM and AN, and E is the point of intersection of IN and AM. FS is drawn perpendicular to AN, and ES is drawn perpendicular

to AM. FS and ES intersect in S. Prove that the acceleration of C is given by ω^2NS, where ω is the uniform angular velocity of AB. [C.U.]

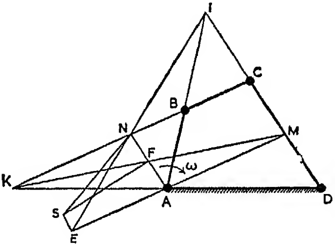

FIG. 72.

17. A point P is moving along a line OP which is turning in the plane of the paper about O (Fig. 73). Referred to fixed rectangular axes OX and OY, the co-ordinates of the point P are x and y. Denoting the length OP by r and the angle POX by θ, then

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

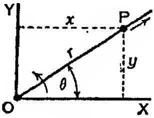

FIG. 73.

Show by differentiation that

$$\frac{d^2x}{dt^2} = -r\left\{\frac{d^2\theta}{dt^2}\sin\theta + \left(\frac{d\theta}{dt}\right)^2\cos\theta\right\} + \frac{d^2r}{dt^2}\cos\theta - 2\frac{dr}{dt}\frac{d\theta}{dt}\sin\theta,$$

$$\frac{d^2y}{dt^2} = r\left\{\frac{d^2\theta}{dt^2}\cos\theta - \left(\frac{d\theta}{dt}\right)^2\sin\theta\right\} + \frac{d^2r}{dt^2}\sin\theta + 2\frac{dr}{dt}\frac{d\theta}{dt}\cos\theta,$$

then, by resolving these accelerations along and perpendicular to the line OP, show that

Acceleration of P along OP is $\dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2$,

Acceleration of P perpendicular to OP is $r\dfrac{d^2\theta}{dt^2} + 2\dfrac{dr}{dt}\dfrac{d\theta}{dt}$.

18. A body moving with plane motion has its instantaneous centre at O and its centre of zero acceleration at Z (Fig. 74).

If the direction of the acceleration of a point P is perpendicular to OP, that is, if the acceleration is in the same direction as the velocity, show that all points such as P lie on a circle passing

through O, P, and Z; also show that the acceleration of O is along the diameter OD_1.

If the direction of the acceleration of a point Q is along QO, that is, if the acceleration is in the direction perpendicular to the velocity, show that all points such as Q lie on a circle passing through O, Q, and Z.

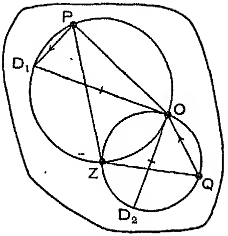

FIG. 74.

Show that the diameters OD_1 and OD_2 of the two circles are mutually perpendicular and that the points D_1, Z, and D_2 lie in a straight line.

19. The crank CB of the slider-crank mechanism ABC (Fig. 75) is rotating about C with uniform angular velocity ω. From the instantaneous centre O of the rod AB, the line OK is drawn parallel to AC to intersect AB produced at K, then KP is drawn parallel to OA to intersect BC produced at P. A circle is drawn through the points O, A, and

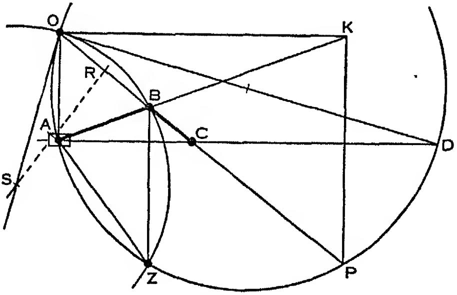

FIG. 75.

P, and then the diameter OD is drawn. Next, OS is drawn perpendicular to OD, and OB is bisected at right angles by the line RS. The lines OS and RS intersect at S. With centre S and radius SO a circle is drawn through the points O and B, intersecting the circle OAP again at Z.

Prove that $BC.BP = OB^2$ and then show that the point P, considered as a point attached to the rod AB, has no acceleration in the direction PC. (*Hint.*—Considering the points P, B, and C, write down the total acceleration of the point P and then find the condition which makes the component in the direction PC zero.)

Show that the point Z is the centre of zero acceleration of the rod AB, and that the acceleration of the slider A may be expressed as $\omega^2 \dfrac{BC}{ZB} \times ZA$.

Finally, show that the point Z could also be obtained by drawing the circle OAP and a circle passing through the points A, B, and C.

CHAPTER V

FORCE, TORQUE, WORK, AND ENERGY

38. **Newton's Laws of Motion.**—*First Law.*—Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by force to change that state.

Second Law.—Rate of change of momentum is proportional to the impressed force and takes place in the direction of the straight line in which the force acts.

Third Law.—To every action there is always an equal and opposite reaction.

39. **Force and Acceleration.**—Denoting force, mass, velocity, and time by P, M, v, and t, respectively, then *momentum*, the product of mass and velocity, is Mv and from Newton's second law of motion

$$P \propto \frac{d}{dt}(Mv)$$

or
$$P = C\frac{d}{dt}(Mv),$$

where C is a constant, or, using suitable units,

$$P = \frac{d}{dt}(Mv).$$

When M is constant, $P = M\dfrac{dv}{dt} = Mf,$

where dv/dt or f is the acceleration,

or
$$\text{Force} = \text{Mass} \times \text{Acceleration}.$$

If s is displacement, $v = ds/dt$ and acceleration may be written as

$$\frac{d^2s}{dt^2} = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}.$$

Also it is sometimes convenient to write mass M as W/g, where W is the weight of the mass and g is the acceleration due to gravity.

Therefore $$P = Mf = \frac{W}{g}\frac{dv}{dt} = \frac{W}{g}v\frac{dv}{ds}$$

are some of the different ways of expressing the same equation.

The dimensions of force in terms of mass M, length L, and time T are ML/T^2 or MLT^{-2}.

Units.—A force of 1 *pound-weight* is the attraction which the earth exerts *in the latitude of London* on a certain standard piece of platinum whose mass is described as being 1 *pound.* Another unit of force is the *poundal,* and 32·2 poundals are approximately equal to 1 pound-weight.

A force of 1 poundal will give a mass of 1 pound an acceleration of 1 foot per second per second.

Therefore if a mass of M pounds is acted on by a force of P poundals and the acceleration is f feet per second per second, the equation of motion is

$$P = Mf.$$

A force of 1 pound-weight will give a mass of 1 pound an acceleration of 32·2 feet per second per second, correct to three figures, and this acceleration is the value of g, the acceleration due to gravity, *in the latitude of London.* It follows that a force of 1 pound-weight will give a mass of 32·2 pounds an acceleration of 1 foot per second per second.

Therefore if a mass of M pounds is acted on by a force of P pounds-weight and the acceleration is f feet per second per second, the equation of motion is

$$P = \frac{M}{32{\cdot}2}f \quad \text{or} \quad 32{\cdot}2\ P = Mf.$$

This is often written as $Pg = Mf$, which, strictly, is incorrect. The value of g varies slightly from place to place, whereas the units of force (as defined above), mass, and acceleration do not depend on position.

If a new unit of mass equal to 32·2 pounds is chosen and if M_s is the number of these units in the mass, then the equation becomes

$$P = M_s f.$$

This larger unit of mass is sometimes called a *slug*.

Suppose now a mass of M units, of W pounds-weight, is acted on by a force equal to its own weight, then the acceleration will be g feet per second per second and the equation may be written as $W = Mg$ if the units of M are suitably selected. Solving for M gives $M = W/g$ and it is evident that mass may be replaced by the ratio of its weight to the acceleration due to gravity. Therefore if the equation $P = Mf$ is written as

$$P = \frac{W}{g} f,$$

the force P will be in pounds-weight, provided the weight W is in the same units and the units of f are the same as those of g.

Much controversy has ranged over the question of writing W/g for M and probably will continue for all time, but the method is correct and very convenient. Most of the American books appear to use W/g, although it is not always clear whether force/acceleration or mass/number is intended. This discussion may seem unimportant to the student, but it is essential that he should know what he is doing.

The advantages of using W/g and regarding it as force/acceleration are that a unit check may be used in an equation and that the use of different units is facilitated.

For example, considering $P = \frac{W}{g} f$, which is the simplest equation in which mass appears:—

(1) With lb.-weight, foot and second units,

$$\text{lb.-wt.} = \frac{\text{lb.-wt.}}{\text{ft./sec.}^2} \times \text{ft./sec.}^2,$$

or $$\text{lb.-wt.} = \text{lb.-wt.}$$

(2) With ton-weight, inch and second units,

$$\text{tons-wt.} = \frac{\text{tons-wt.}}{\text{inches/sec.}^2} \times \text{inches/sec.}^2,$$

or $$\text{tons-wt.} = \text{tons-wt.}$$

Usually an engineer talks of a force of so many pounds when he means pounds-weight, but it is doubtful whether this ever leads to confusion.

40. Torque and Angular Acceleration.—Consider a small mass m travelling, with velocity v at time t, in a circular path of radius $OA = r$ and acted on by a tangential accelerating force P (Fig. 76).

Since $P = \frac{d}{dt}(mv)$, $Pr = \frac{d}{dt}(mrv)$.

But $v = \omega r$ where ω is the angular velocity of the radius OA, therefore

$$Pr = \frac{d}{dt}(mr^2\omega).$$

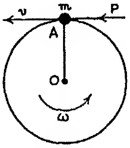

FIG. 76.

If a body consisting of a large number of particles of mass m is rotating with angular velocity ω at time t, each particle being acted on by a tangential force P, then, using the symbol Σ to denote the words "the sum of,"

$$\Sigma Pr = \frac{d}{dt}\left\{\Sigma(mr^2)\omega\right\},$$

or $$T = \frac{d}{dt}(I\omega),$$

where $T = \Sigma Pr$ denotes the *turning moment* or *torque* on the body and $I = \Sigma mr^2$ is known as the *moment of inertia* of the body (see Art. 44).

The product $I\omega$ is the *angular momentum* of the body, therefore the rate of change of angular momentum of the body is equal to the applied torque. Of course, for this equality to hold good, suitable units must be used.

Since $I\omega = \Sigma(mr^2)\omega = \Sigma(mr^2v/r) = \Sigma mvr$ and mv is a

momentum, Σmvr, or $I\omega$, is the moment of a momentum. Therefore angular momentum is also called *moment of momentum*.

If the moment of inertia I is constant,

$$T = I\frac{d\omega}{dt} = I\alpha,$$

where $d\omega/dt$ or α is the angular acceleration.

Units.—If $I = \Sigma mr^2$ is in pound-feet^2 and α is in rad./sec.^{2}, then T will be in poundal-feet. If mass m is written as w/g and w is in pounds-weight and g is in feet/sec.^{2}, then T will be in pound-weight-feet or, more briefly although strictly incorrectly, pound-feet.

41. **Impulse.**—If a force P acts on a mass M for a time t and changes the velocity of the mass from u to v, then, since

$$P = M\frac{dv}{dt}$$

$$\int_0^t P dt = M\int_u^v dv = M(v - u).$$

If P is a variable and has a mean value P′, then

$$\int_0^t P dt = P't \quad \text{and} \quad P't = M(v - u).$$

The product $P't$ is called an *impulse* and is equal to the change of momentum it produces. When t is indefinitely small, as in the case of a blow or collision, P′ is very large and is called an *impulsive force*. Generally in such a case it is not possible to measure t accurately and therefore P′ cannot be determined.

In a similar way it can be shown that

$$\int_0^t T dt = T't = I(\omega_2 - \omega_1),$$

where T′ is the mean value of a variable torque T acting for a time t, $\omega_2 - \omega_1$ is the change of angular velocity, and $I(\omega_2 - \omega_1)$ is the change of angular momentum. The

product $T't$ is the impulse of the torque and is equal to the change of angular momentum it produces. When t is indefinitely small, T' is very large and is called an *impulsive torque* or *impulsive couple.*

42. Conservation of Momentum—Linear Momentum.—*If, in any direction, the sum of the components of the external forces acting on a system of bodies is zero, the total momentum of the system is constant in that direction.* For example, if two bodies act on one another with a force P for a time t, the impulse is Pt, and if there is no external force acting in the direction of the force P, the momentum gained by one is equal to that lost by the other, therefore the total momentum of the system is unchanged.

Angular Momentum or Moment of Momentum.—*If, in a system of bodies, the sum of the moments of the external forces about any fixed axis is zero, the angular momentum of the system about that axis is constant.* For example, if two discs rotating about a common axis act on one another with a torque T for a time t, and there is no external torque acting on the discs, the angular momentum gained by one is equal to that lost by the other and the total angular momentum of the system is unchanged.

43. Centrifugal Force.—When a body of mass m moves in a circular path, an inward radial acceleration has to be provided and its value is v^2/r or $\omega^2 r$ (Art. 28, p. 47).

Since $P = mf$, the inward radial force which is required in order to produce the inward radial acceleration is $P = mv^2/r = m\omega^2 r$ and this is called the *centripetal force.* The mass resists this inward force with an equal and opposite outward force called the *centrifugal force,* but this force does *not* act on the mass. If centripetal force ceases to act, for example if the mass is travelling in a circle at the end of a radial string on a horizontal surface and the string is suddenly cut, the centrifugal force disappears too and the mass goes off in a tangential direction.

It is common practice, however, *to regard centrifugal force as an outward radial force acting on the mass* and then to treat the forces acting on the mass as being in equilibrium.

Centrifugal force regarded in this way is merely a particular case of a reversed effective force or reversed accelerating force (Art. 74, p. 132). This is an artifice which produces the desired result, but it is important to understand that it is only an artifice.

The following example on a conical pendulum shows that either centripetal force or the artificial centrifugal force may be used to produce identical results.

Consider the conical pendulum shown in Fig. 77 where a string OA supports a body of mass m and weight W, which is moving in a horizontal circle of radius r about a vertical axis OY. Let the angular velocity be ω and let the angle YOA be θ. It is required to find ω in terms of W, θ, m, and r.

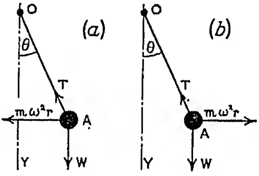

FIG. 77.

The horizontal component of the tension T in the string is T sin θ and this component is equal to and provides the centripetal force $m\omega^2 r$ shown at (a), that is

$$T \sin \theta = m\omega^2 r.$$

Also, resolving vertically, in which direction there is no accelerating force,

$$T \cos \theta = W.$$

From these equations, $\omega = \sqrt{\dfrac{W \tan \theta}{mr}}$.

Using centrifugal force and referring to the Fig. at (b), $m\omega^2 r$ is regarded as a radial force acting outwards on the mass, and all the forces acting on the mass are treated as being in statical equilibrium.

Resolving horizontally,

$$T \sin \theta = m\omega^2 r,$$

and resolving vertically,

$$T \cos \theta = W.$$

These two equations are identical with those arrived at when centripetal force was used, thus the two methods produce the same results.

44. Moment of Inertia.—If m is the mass of a particle of a body at a perpendicular distance r from an axis through O perpendicular to the paper (Fig. 78), then the *moment of inertia of the body* about the axis through O is defined as $I = \Sigma mr^2$, where Σ denotes "the sum of" and the summation includes all the particles of the body.

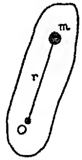

FIG. 78.

If M is the total mass of the body and k is such that

$$I = Mk^2 = \Sigma mr^2 \quad \text{or} \quad k = \sqrt{(I/M)},$$

then k is defined as the *radius of gyration* of the body about the axis through O.

The *moment of inertia of an area* is defined in a similar manner by substituting area for mass. If, in Fig. 78, a is an element of area at a distance r from O and the whole figure has an area A,

$$I = Ak^2 = \Sigma ar^2 \quad \text{and} \quad k = \sqrt{(I/A)}.$$

Sometimes the moment of inertia of an area is called the *second moment of an area*, which is a more descriptive phrase since an area, having no mass, cannot have inertia. However, the term moment of inertia is more usual in practice.

Units.—For a mass, since $I = Mk^2$, the units of I will depend on those of M and k. If M is in pounds and k is in feet, then I will be in pounds $\times$ feet^2. If M is expressed as W/g, force/acceleration, with say W in pounds-weight and g in feet per second per second, and if k is in feet, then I will be in $\dfrac{\text{lb.-wt.}}{\text{ft./sec.}^2} \times \text{ft.}^2$ or lb.-wt. $\times$ ft. $\times$ sec.^{2}.

For an area, since $I = Ak^2$, the units of I will be those of length^4; if length is in inches then I will be in inches^4.

45. Moment of Inertia—Theorems.—Two theorems and their corollaries are given below.

Theorem I.—If I_x and I_y are the moments of inertia of a plane figure (Fig. 79) about axes OX and OY in its plane and perpendicular to one another, and if I_z is the moment of inertia of the figure about an axis OZ perpendicular to the plane XOY, then $I_z = I_x + I_y$. I_z is called a *polar moment of inertia.*

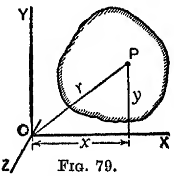

FIG. 79.

Consider a small element P, of area a, whose distances from OY, OX, and O are x, y, and r, respectively, then $r^2 = x^2 + y^2$, $ar^2 = ax^2 + ay^2$, and $\Sigma ar^2 = \Sigma ax^2 + \Sigma ay^2$, therefore $I_z = I_y + I_x$.

Corollary 1.—If OZ is a fixed axis perpendicular to the plane of the figure and if OX and OY are any two mutually perpendicular axes in the plane, then $I_x + I_y$, being equal to I_z, is constant.

Corollary 2.—Since $I_x + I_y$ is constant, it follows that if I_x is a maximum, I_y is a minimum.

Theorem II.—Let I be the moment of inertia of a surface or body about an axis XX passing through its centre of gravity G (Fig. 80), and let I_1 be the moment of inertia of the surface or body about an axis X_1X_1 parallel to XX and at a perpendicular distance r from it. Let A be the area of the surface and M the mass of the body. It can be shown (see D. A. Low's *Applied Mechanics*, Art. 68, p. 51) that—

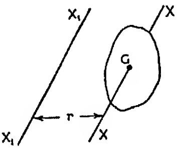

FIG. 80.

For the surface $\quad I_1 = I + Ar^2$.

For the body $\quad I_1 = I + Mr^2$.

Corollary 1.—If k and k_1 are the radii of gyration about the axes XX and X_1X_1, respectively, $I = Ak^2$ or Mk^2 and $I_1 = Ak_1^2$ or Mk_1^2. Hence $k_1^2 = k^2 + r^2$.

Corollary 2.—The radius of gyration about a given axis passing through the centre of gravity is less than the radius of gyration about an axis parallel to the given axis, and the axis about which the radius of gyration is least must pass through the centre of gravity.

46. Moment of Inertia—Routh's Rule.—

Moment of inertia about an axis of symmetry

$$= mass \times \frac{sum\ of\ squares\ of\ perpendicular\ semi\text{-}axes}{3,\ 4,\ or\ 5}.$$

The denominator is to be 3, 4, or 5, according as the body is rectangular, elliptical, or ellipsoidal.

In this rule the word mass may be interpreted as either mass or area. Also it is to be understood that the semi-axes are perpendicular to the axis about which the moment of inertia is required. For instance, for a circle (this is the particular case of an ellipse where the major and minor axes are equal) there is one semi-axis if the moment of inertia is about a diameter, and there are two semi-axes if it is about an axis through the centre perpendicular to the plane of the figure.

In the case of right solids, all sections perpendicular to the axis about which the moment of inertia is required must be either rectangular, circular, or elliptical and must all be equal.

The rule cannot be used to find the moment of inertia of a square about a diagonal, but this is the same as the moment of inertia about an axis through the centre and parallel to one side. The proof follows at once from Art. 45, Theorem I, Corollary 1.

47. Moment of Inertia.—*Example.*—To find the moment of inertia and radius of gyration of a right circular cylinder, of radius R, length l, and mass M, about the longitudinal axis.

Using Routh's rule, there are two perpendicular semi-axes, each equal to R, therefore

$$I = M\frac{R^2 + R^2}{4} = M\frac{R^2}{2}$$

and $$k = \sqrt{(I/M)} = R/\sqrt{2}.$$

Working from first principles, let m be the mass per unit volume, then $m = M/\pi R^2 l$.

For a thin cylinder of radius r and thickness δr (Fig. 81) the mass is $2\pi r\delta r l m$ or $2r\delta r \mathrm{M}/\mathrm{R}^2$ and the moment of inertia is $2r^3\delta r\mathrm{M}/\mathrm{R}^2$.

The total moment of inertia is

$$\mathrm{I}=\frac{2\mathrm{M}}{\mathrm{R}^2}\int_0^{\mathrm{R}} r^3 dr=\frac{2\mathrm{M}}{\mathrm{R}^2}\frac{\mathrm{R}^4}{4}=\mathrm{M}\frac{\mathrm{R}^2}{2}$$

and, as before, $k=\mathrm{R}/\sqrt{2}$.

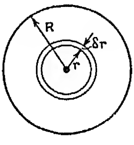

FIG. 81.

Other examples of moments of inertia are given in the exercises at the end of this Chapter, in D. A. Low's *Applied Mechanics*, p. 52, and in the author's *Mathematics*, p. 252.

48. **Work.**—Work is measured by the product of force and the distance through which it acts. In Fig. 82 force P is plotted against displacement s. The area of the shaded strip of width δs represents $\mathrm{P}\delta s$, the work done during the small displacement δs. The area under the curve between the ordinates at A and B represents the work done whilst the displacement increases from $s=s_1$ to $s=s_2$, therefore

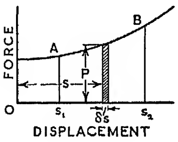

FIG. 82.

$$\text{Work between these limits}=\int_{s_1}^{s_2}\mathrm{P}ds.$$

If the equation to the curve AB is known, the value of this integral may usually be obtained. If the equation is unknown, the actual area under the curve may be found graphically, and then, taking into account the force and displacement scales, the work done may be evaluated. Suppose 1 inch horizontally represents a feet and 1 inch vertically represents p pounds, then 1 square inch represents ap foot-pounds. Hence, if the whole area considered is A square inches, the work done will be apA foot-pounds.

From what has been said about units of force, it follows that p should be in pounds-weight instead of pounds, and then, if displacement is in feet, the work done would be in foot-pounds-weight. However, in practice "weight"

is omitted and the work is said to be in foot-pounds. Other commonly used units are inch-pounds, foot-tons, and inch-tons.

49. Work in Raising a System of Masses.—When a number of masses are raised through different heights, or when all the parts of one mass are not raised through the same height, the amount of work done is obtained by multiplying the total weight lifted by the distance through which the centre of gravity of the system is raised (Proof is given in D. A. Low's *Applied Mechanics*, p. 24).

50. Work done by a Torque.—Suppose a uniform torque T pound-feet acts through an angle of θ radians (Fig. 83). Let T be replaced by a force P pounds acting at a radius r feet, the product Pr being equal to T. The force P is displaced a distance $s = r\theta$ and the work done is

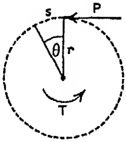

Fig. 83.

$$Ps = Pr\theta = T\theta \text{ foot-pounds.}$$

If the torque starts from zero and increases uniformly to the value T, then the mean value is $\frac{1}{2}$T and the work done is $\frac{1}{2}T\theta$. In general, if a varying torque T acts through an angle θ,

$$\text{Work done} = \int_0^\theta T d\theta.$$

If the equation between T and θ is known, the value of this integral may usually be obtained, otherwise graphical integration must be used as explained in Art. 48.

51. Energy.—Energy is the capacity for doing work.

Potential Energy is energy due to position or configuration. If a body of weight W is lifted a height h from the ground, the work done is Wh and this is stored up as potential energy due to the position of the body. If the body is then allowed to fall a distance h it can be made to do work equal to Wh. A deflected spring has potential energy stored in it, for it can be made to do work in returning to its unstrained position.

Kinetic Energy is energy due to motion. If a body of weight W which has been lifted through a height h is allowed to fall, it will lose potential energy and gain kinetic energy. For instance, when it has fallen a distance $\frac{1}{4}h$ its potential energy will be $\frac{3}{4}Wh$ and its kinetic energy will be $\frac{1}{4}Wh$. Just as it reaches the ground its potential energy will be zero and its kinetic energy will be Wh.

52. Conservation of Energy.—Energy is indestructible but may change its form. For example, work done against friction will generate heat. If there is no friction and no external work is done, then the total energy in any mechanical system is constant.

In Mechanics, the conservation of energy is stated by the relation

Potential Energy + Kinetic Energy = Constant.

53. Kinetic Energy of Translation and Rotation.—Suppose a body starts from rest and acquires a velocity v in time t. It is required to find the kinetic energy.

If the mass is M, the displacement is s and the accelerating force is P, then

$$P = M\frac{dv}{dt} = Mv\frac{dv}{ds}.$$

Therefore $$\int_0^s P\,ds = M\int_0^v v\,dv = \tfrac{1}{2}Mv^2,$$

and this is the kinetic energy. It follows that if the velocity of the body changes from v_1 to v_2, the change of kinetic energy is $\frac{1}{2}M(v_2^2 - v_1^2)$.

The kinetic energy of a rotating body is found in a similar way. Suppose the moment of inertia about the axis of rotation is I and that at time t the torque is T, the angle turned through from rest is θ and the acquired angular velocity is ω.

Since $$T = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\,\frac{d\theta}{dt} = I\omega\frac{d\omega}{d\theta},$$

therefore $$\int_0^\theta T\,d\theta = I\int_0^\omega \omega\,d\omega = \tfrac{1}{2}I\omega^2$$

and this is the kinetic energy due to rotation. If the angular velocity changes from ω_1 to ω_2, the change of kinetic energy is $\frac{1}{2}I(\omega_2^2 - \omega_1^2)$.

If a body has motions of translation and rotation at the same time, the linear velocity of the axis of rotation being v and the angular velocity about the axis of rotation being ω, then the total kinetic energy is

$$\tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2.$$

54. Power.—Power is the rate of doing work. If a force P is working at a speed V, then the work done per unit time is PV.

Horse-power is the name given to the particular rate of 33,000 foot-pounds per minute or 550 foot-pounds per second. If a force of P pounds is doing work at the rate of V feet per minute, then

$$\text{Horse-power} = \frac{PV}{33{,}000}.$$

If a torque of T pound-feet is turning a shaft at N revolutions per minute, the angle turned through per minute is $\theta = 2\pi N$ radians, the work done per minute is $T\theta = 2\pi NT$,

and $$\text{Horse-power} = \frac{2\pi NT}{33{,}000}.$$

55. Vector Representation of Angular Velocity and Torque.—Consider a disc C or any other body rotating about an axis AB (Fig. 84) with an angular velocity ω. If the direction of rotation is clockwise when viewed from A, the angular velocity may be represented in magnitude and sense by a line *oc* drawn parallel to the axis AB, to a suitable scale, the sense being determined, by using the right-handed screw convention, as from *o* to *c*, the direction in which a right-handed screw would advance when turned clockwise in a fixed nut. Viewed from B, the direction of

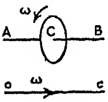

FIG. 84.

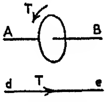

FIG. 85.

rotation is anticlockwise, but a right-handed screw would still move in the direction AB if turned anticlockwise, therefore *oc* still represents ω. If the direction of ω is reversed, then ω is represented by *co*.

What has been said about angular velocity may also be applied to angular momentum, because the latter is a constant times velocity.

In the same way a torque T (Fig. 85) may be represented by a line *de* drawn parallel to the axis AB, to a suitable scale.

56. Gyrostatic Motion.—In Fig. 86, OX, OY, and OZ are mutually perpendicular axes, and a disc C is rotating about OX with a constant angular velocity ω, the direction being clockwise when viewed from O. Suppose now that a torque T is applied to the disc, in a clockwise direction when looking from Z to O. If the diameter MN is parallel to OZ, the torque is clockwise about this diameter when viewed from N. It is required to investigate the motion of the disc due to the application of the torque T.

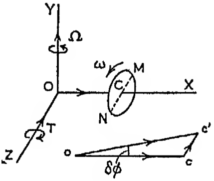

FIG. 86.

Let I be the moment of inertia of the disc about OX, then the angular momentum about OX is $I\omega$. Draw the vector *oc* parallel to OX, to a suitable scale, to represent $I\omega$. Now a torque T applied for a time δt will produce a change of angular momentum equal to $T\delta t$, therefore draw *cc'* parallel to ZO to represent $T\delta t$ and join *oc'*. Let the angle *coc'* be $\delta\phi$. The triangle *coc'* is in the ZOX plane and the angle *occ'* represents 90°.

Since ω is a constant angular velocity, *oc'* must also represent $I\omega$. As $\delta\phi$ is very small, *cc'* is approximately equal to $oc\delta\phi$, therefore

$$T\delta t = I\omega\delta\phi \quad \text{or} \quad T = I\omega\frac{\delta\phi}{\delta t}, \quad \text{approximately.}$$

When δt is made indefinitely small and approaches zero,

$$T = I\omega \frac{d\phi}{dt} = I\omega\Omega,$$

where $\Omega = \frac{d\phi}{dt}$ is the angular velocity in the ZOX plane.

Therefore it is seen that instead of the axis OX turning about OZ, due to the torque T, it turns about the axis OY in an anticlockwise direction when viewed from Y. This motion of the axis of spin OX, about the axis OY, is called *precession* and the rotating disc *precesses*.

It is to be noted that the precession *tends* to cause the axis of spin to move round so that it coincides with the axis of the applied torque and so that the spin and the torque would then have the same sense. Actually, of course, when the axis of spin moves round, the axis of torque also moves round because the angle between these axes remains a right angle.

When a torque is applied in such a way as to cause a spinning body to precess, the motion is called *gyrostatic*.

Exercises V

1. A body, weighing 100 lb. and supported on a horizontal plane, is moved by a horizontal force of 20 lb. against a constant frictional force of 15 lb. If the body starts from rest, find the value of its acceleration ar [illegible] of 15 seconds. If the 20-lb. force is then [illegible] .me will elapse before the body comes to rest?

2. A pile-driver hammer weighing 1500 lb. falls a distance of 2·5 feet on to an inelastic pile weighing 1000 lb. and drives it 1·5 inches into the ground against a uniform resistance. Find the velocity with which the pile begins to move and the time during which it is moving. Find also the value of the ground resistance (*a*) neglecting the weights of the hammer and the pile, (*b*) taking these weights into account.

3. Two masses M and m, moving along a straight line in the same direction with velocities U and u respectively, collide and continue their motion with a common velocity v. Show that the loss of kinetic energy is

$$\frac{Mm(U-u)^2}{2(M+m)}.$$

4. A body, initially at rest, is acted on by a force which varies as the square of the time and is 50 lb. after 20 sec. Find the time-average and the space-average of the force during the first 20 sec.

5. What would be the answers to the preceding exercise if the force varies as the time and is 100 lb. after 20 sec.?

6. A car weighing 1 ton starts from rest on a level road. The difference between the tractive effort and the resistance to motion is initially 500 lb. and falls, the decrease being proportional to the distance travelled, until it is zero at the end of 250 yards. Find the maximum speed in miles per hour.

7. If a car weighing 1 ton starts from rest on a level road and the accelerating force in pounds is $500-0{\cdot}045v^2$, where v is the speed in miles per hour, find the speed at the moment when the car has travelled 250 yards.

8. A shell weighing 2000 lb. is fired horizontally with a velocity of 2100 ft. per sec. from a gun mounted on a truck, the combined weight of gun and truck being 170 tons. If the truck is standing on an incline of 1 in 5, measured as a sine, find the distance it will travel up the slope.

9. If in the preceding exercise a constant force of F tons is acting down the incline during the motion, find its value so that the truck may come to rest after travelling 5 feet.

10. Two men exerting together a force of 90 lb. weight put a railway wagon into motion. The wagon weighs 6 tons and the resistance to motion is 10 lb. per ton. How far does the wagon advance in 1 minute; and at what rate, in horse-power, are the men working at the end of the minute?

If the men can at most do work at the rate of 0·8 horse-power, at what constant speed can they keep the wagon moving? [C.U.]

11. A flywheel is secured to a shaft which is supported horizontally in bearings, and a weight W is carried by a taut string attached to and wrapped round the shaft (Fig. 87). Starting from rest, the weight W descends a distance of 4 feet, then the string is detached from the shaft and the latter does a further 150 turns before coming to rest. Assuming that a constant resisting couple C acts on the shaft during the whole experiment, find its value in lb.-inches.

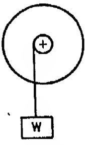

Fig. 87.

Weight of flywheel and shaft is 80 lb. and their radius of gyration is 10 inches. Diameter of shaft is $1\frac{1}{2}$ inches and W = 15 lb.

12. A flywheel is fixed on a shaft, 1·5 inches in diameter, which is supported on two parallel rails having a slope of 1 in 10 (measured as a sine). If the flywheel and shaft start from rest and in 10 seconds roll 6·5 feet down the slope without

slipping, find their radius of gyration about the centre line of the shaft.

13. For starting a small diesel engine by hand, all compression is relieved while the handle is swung until a speed of 100 r.p.m. is attained. Full compression is then brought in, and the energy stored in the flywheel is relied upon to move the piston past top dead centre at an instantaneous crankshaft speed of 50 r.p.m. If full compression starts at 40° after bottom dead centre, and the work done during compression is given by $\int \frac{C \sin \theta}{1 - \cos \theta} d\theta$, where C = 125 lb.-ft., calculate the radius of gyration of the flywheel whose weight is 2 cwt.

14. A body weighing 2000 lb., starting from rest, is propelled in a straight line against a constant resistance of 40 lb. by a constant horse-power which is equal to 1·2. Find the time taken for the body to reach a speed of 10 ft. per sec.

15. A car has a speed of 60 m.p.h. round a curve on a banked track of 100 yards radius. If there is no lateral force between the car and the track, find [illegible].

If a car weighing 2000 lb. [illegible] what would be the lateral force between the car and the track?

16. A railway track of 4 ft. $8\frac{1}{2}$ in. gauge is laid in a curve of 3000 ft. mean radius. Find the super-elevation necessary if a train travelling at 70 m.p.h. is to exert the same pressure on the outer rail as it will on the inner rail when travelling at 30 m.p.h.

17. A liner of 50,000 tons displacement is driven by engines which are capable of maintaining a propulsive force of 250 tons at all speeds. Resistance to motion varies as the square of the speed and the vessel reaches a steady speed at 30 m.p.h. Find the horse-power developed at this speed, and if the engines are then shut off, find the time which elapses before the speed has fallen to 15 m.p.h. [C.U.]

18. Water is pumped at the rate of 150 tons per hour through a pipe to a height of 40 feet where it is delivered through a nozzle of 3 inches diameter. Find the horse-power required to drive the pump if 35 per cent. of the energy supplied is wasted in frictional resistances. Take the weight of a cubic foot of water as 62·3 lb.

19. A railway truck weighing 8 tons, travelling at 4 m.p.h. on a level track, collides with a stationary truck weighing 12 tons. When the spring buffers are in contact and the distance between the trucks is reduced by x inches the reaction is $0{\cdot}85x$ tons. Find the common velocity which the trucks have for an instant, the loss of kinetic energy, and the compression in inches of the buffer springs.

20. A motor-car weighing 2000 lb. exerts 18 horse-power when travelling at a uniform speed of 20 m.p.h. up a slope of 1 in 7 (measured as a sine). Find (*a*) the horse-power exerted in overcoming gravity, and (*b*) the resistance due to friction, air pressure, etc.

21. On the level the maximum speed of a train of total weight W tons is V miles per hour, the resistance to the motion when the speed is v miles per hour being $(a+bv)$ tons weight, where a and b are constants. The pull of the engine is constant and equal to P tons weight. Show that the maximum horse-power of the engine when the train is on an up-gradient of angle α, under the same resistance, is

$$\frac{448}{75}\mathrm{PV}\left(1-\frac{\mathrm{W}\sin\alpha}{b\mathrm{V}}\right).$$

Calculate the value of V, given that the resistances corresponding to speeds of 5 and 25 m.p.h. are respectively 1 and $3\frac{1}{2}$ tons weight, that $\mathrm{W}=450$, that $\sin\alpha=1/200$, and that the values of the maximum horse-power developed on the level and on the incline are respectively 360 and 252. [C.U.]

22. A horizontal shaft in fixed bearings carries an eccentric of radius a, the distance of whose centre from that of the shaft is c. The eccentric bears against the horizontal surface of a bar carrying a weight. The bar is so constrained that it is only free to move vertically, remaining parallel to itself. The shaft is slowly turned so that the bar rises and falls. Find an expression for the couple which must be applied to the shaft in terms of the angle through which it has been turned. The weight of the eccentric may be neglected, and the only friction to be taken account of is that between the bar and the eccentric, the coefficient for which is μ.

If $\mu=\frac{1}{4}$ and $a=2c$, prove that the efficiency of the arrangement, when used as a machine for raising the bar and weight from the lowest to the highest position by a half-turn of the shaft is approximately 0·56. [C.U.]

23. An aeroplane travelling at a speed V relative to the air experiences a resistance $\mathrm{R}=a\mathrm{V}^2+b/\mathrm{V}^2$, where a and b are constants within certain limits of V. Show that, within these limits of V, the power absorbed in air resistance has a minimum value H_0, at a speed V_0, where

$$\mathrm{H}_0=4\left(\frac{ab^3}{27}\right)^{\frac{1}{4}}, \qquad \mathrm{V}_0^4=b/3a.$$

Assuming that the effective thrust power Z of the propeller is independent of V, find the greatest rate of gain of height, and

show that the aeroplane is then climbing at an angle $\sin^{-1}\dfrac{Z-H_0}{WV_0}$ to the horizontal, where W is the weight of the aeroplane. [C.U.]

24. Let it be assumed that the effective rate of working of the engine of a motor-car is represented by the expression $hn(n_1-n)$ where n is the number of revolutions per second and h, n_1 are certain constants; also that the velocity v is equal to kn where k is a constant. If there is a fixed resistance to the motion of the car equal to λ times its weight (w), find the speed with which the car will ascend an incline of angle a. If the value of k is capable of adjustment, show that to secure the best speed k must be equal to

$$hn_1/2w(\lambda+\sin a),$$

the best speed being

$$hn_1^2/4w(\lambda+\sin a).$$

If the maximum power of the engine is 20 H.P., the weight of the car is one ton, and the best speed up a slope of one in ten is 20 miles per hour, show that the best speed on the level is nearly 50 miles per hour and find the best speed up a slope of one in five. [C.U.]

25. A particle of mass m moves in a vertical plane in a medium whose resistance is km multiplied by the velocity. Show that the component velocities referred to horizontal and vertical axes are

$$\dot{x}=Ae^{-kt}, \qquad \dot{y}=Be^{-kt}-\frac{g}{k},$$

and that the resultant acceleration is in a fixed direction. Prove also that the vertical distance of the particle from a line through the point of projection parallel to the direction of the resultant acceleration is gt/k. [C.U.]

26. A motor-car is travelling round a left-hand bend of 20 feet mean radius at a speed of 15 m.p.h. The flywheel and crank-shaft rotate at 1500 r.p.m. in a clockwise direction when viewed from the front; their weight is 100 lb. and their radius of gyration is 7·5 inches. The distance between the front and back axles is 8 feet. Find the change in the pressures on the wheels due to gyrostatic action of the flywheel and crankshaft.

27. A four-wheeled vehicle is rounding a curve of 200 feet mean radius at a speed of 30 m.p.h. The wheel track is 4 feet 6 inches and each wheel has a diameter of 2·5 feet, a radius of gyration of 1 foot, and weighs 50 lb. Find the alteration of pressure between each wheel and the track, due to gyrostatic action.

Moments of Inertia

In the exercises which follow, show that the values of the moments of inertia are as given.

28. A straight and uniform slender rod, of length l and mass M, about an axis perpendicular to it and passing through one end, $I = \frac{1}{3}Ml^2$.

If the axis passes through the centre of the rod instead of through one end, $I = \frac{1}{12}Ml^2$.

29. Rectang[illegible] base of length b and height h, about an axis [illegible] base, $I = \frac{1}{3}bh^3$.

If the axis passes through the centre of gravity and is parallel to the base, $I = \frac{1}{12}bh^3$.

If the axis passes through the centre of gravity and is perpendicular to the plane of the rectangle, $I = \frac{1}{12}bh(b^2 + h^2)$.

30. A square, length of side a, about a diagonal, $I = \frac{1}{12}a^4$.

31. Triangle, base of length b and altitude h, about an axis coinciding with the base, $I = \frac{1}{12}bh^3$.

If the axis passes through the centre of gravity of the triangle and is parallel to the base, $I = \frac{1}{36}bh^3$.

If the axis passes through the vertex of the triangle and is parallel to the base, $I = \frac{1}{4}bh^3$.

32. Circle of radius R, about an axis passing through its centre an[illegible] its plane, $I = \frac{1}{2}\pi R^4$.

[illegible] $I = \frac{1}{4}\pi R^4$.

33. A solid cylinder of length l, radius R and mass M, about a diameter at one end, $I = M(\frac{1}{4}R^2 + \frac{1}{3}l^2)$.

34. A solid sphere of radius R and mass M, about a diameter, $I = \frac{2}{5}MR^2$. (It is known that the volume of a sphere is $\frac{4}{3}\pi R^3$.)

35. A solid right circular cone, about a diameter in the base, taking R as the base radius, H as the altitude, and M as the mass, $I = \frac{3}{20}M(R^2 + \frac{2}{3}H^2)$.

CHAPTER VI

DIMENSIONS—DYNAMICAL SIMILARITY

57. Dimensions.—In Mechanics most quantities may be expressed in terms of one or more of the fundamental quantities, mass M, length L, and time T, although sometimes it may be convenient to use force F instead of mass M.

A mass is of one dimension in M, a length is of one dimension in L, and a period of time is of one dimension in T. An area being length multiplied by length, or L^2, is of two dimensions in L, and a volume, or L^3, is of three dimensions in L.

Velocity, being length divided by time, or L/T or LT^{-1}, is of one dimension in L and minus one in T. Acceleration, or L/T^2 or LT^{-2}, is of one dimension in L and minus two in T. An angle measured in radians is an arc divided by a radius, or L/L, which is a ratio or mere number and is dimensionless. Therefore angular velocity has the dimension 1/T or T^{-1}, and angular acceleration has the dimensions $1/T^2$ or T^{-2}.

Since force = mass × acceleration, the dimensions of force F in terms of M, L, and T are ML/T^2 or MLT^{-2}.

In terms of F and L, stress or force/area has the dimensions F/L^2 or FL^{-2}, and in terms of M, L, and T this becomes MLT^{-2}/L^2 or $ML^{-1}T^{-2}$.

The dimensions of various quantities are tabulated on p. 90 in terms of M, L, and T and of F, L, and T.

In any equation having a physical meaning and which is true in any system of units, the dimensions of all terms must be the same. This is shown in the examples which follow.

Example 1.—Consider the relation between velocity, acceleration, and time, that is $v = ft$.

Quantity.	Dimensions. In Terms of M, L, T.	Dimensions. In Terms of F, L, T.
Mass	M	$FL^{-1}T^2$
Force	MLT^{-2}	F
Moment of Force	ML^2T^{-2}	FL
Momentum or Impulse	MLT^{-1}	FT
Angular Momentum	ML^2T^{-1}	FLT
Work or Energy	ML^2T^{-2}	FL
Power	ML^2T^{-3}	FLT^{-1}
Moment of Inertia	ML^2	FLT^2
Stress	$ML^{-1}T^{-2}$	FL^{-2}
Density	ML^{-3}	$FL^{-4}T^2$
Viscosity	$ML^{-1}T^{-1}$	$FL^{-2}T$
Kinematic Viscosity	L^2T^{-1}	L^2T^{-1}

The dimensional equation is

$$L/T = (L/T^2)T$$

or $$LT^{-1} = LT^{-1}.$$

Example 2.—Consider the equation $s = ut + \frac{1}{2}ft^2$.

The dimensional equation is

$$L = (L/T)T + (L/T^2)T^2$$

or $$L = L + L.$$

Since numerical coefficients are dimensionless, they do not appear in dimensional equations and therefore a dimensional check is not complete evidence that an equation is correct.

Example 3.—It is required to examine the dimensions of the terms in the equation connecting the loss of potential energy with the gain of kinetic energy when a body rolls down an incline.

The equation is $Wh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$,

where the symbols have the usual meanings.

The dimensional equation is

$$\left(\frac{ML}{T^2}\right)L = M\left(\frac{L}{T}\right)^2 + ML^2\left(\frac{1}{T}\right)^2$$

or $$ML^2T^{-2} = ML^2T^{-2} + ML^2T^{-2}.$$

Example 4.—The relation between the torque T_q and the shear stress f in a shaft of diameter d is $T_q = \pi d^3 f/16$. It is required to check the dimensions of this formula.

This is an example where it is simpler to use F than M.

The dimensional equation is

$$FL = L^3F/L^2$$

or

$$FL = FL.$$

In terms of M, L, and T the dimensional equation is

$$\left(\frac{ML}{T^2}\right)L = L^3\left(\frac{ML}{T^2}\right)\Big/ L^2$$

or

$$ML^2T^{-2} = ML^2T^{-2}.$$

Example 5.—A list of formulæ gave the stress f in a leaf spring as

$$f = \frac{3}{2}\frac{Wl^a}{nbt^2}$$

where W is load, l is length, n is the number of leaves, and b and t are respectively the breadth and thickness of each leaf. The index a was a numerical value which had been altered and made illegible. It is required to find the value of a. [This simple problem actually occurred and the method of dimensions produced the answer in far less time than it could have been obtained in any other way.]

In terms of F and L the dimensional equation is

$$F/L^2 = FL^a/L^3$$

or

$$FL^{-2} = FL^{a-3}.$$

Therefore $\quad a - 3 = -2 \quad$ and $\quad a = 1.$

58. Dimensional Method of Determining Indices in Formulæ.—Sometimes it is convenient to use the method of dimensions to find how one quantity varies in relation to other quantities. The method is shown in the following examples:—

Example 1.—A torque T_q is transmitted through a shaft of diameter d and it is required to use the dimensional

method to find how the torque varies with d and with the shear stress f.

Assume that $$T_q = Cd^a f^b,$$

where C is a constant and a and b are unknown indices. Since a constant is dimensionless, this method will not enable the value of C to be determined. In general, it is incorrect to assume only one term on the right-hand side as it might be necessary to have a number of terms, but this point will be explained further in the next Art. In the present example, and in each of the two which follow, it happens that only one term is required on the right-hand side.

Since $$T_q = Cd^a f^b,$$

the dimensional equation, in terms of F and L, is

$$FL = L^a(F/L^2)^b$$

or $$FL = L^{a-2b}F^b.$$

Equating indices of F, which must be the same on each side of the equation,

$$b = 1.$$

Similarly, equating indices of L,

$$a - 2b = 1,$$

therefore $$a - 2 = 1 \quad \text{or} \quad a = 3.$$

The dimensional equation becomes

$$FL = L^3(F/L^2)$$

and the required equation is

$$T_q = Cd^3 f,$$

showing that the torque is proportional to the stress and the cube of the diameter.

Example 2.—To find how the periodic time t of a pendulum varies with the mass m, the length l, and with g.

Assume $$t = Cm^a l^b g^c,$$

where C is a constant and a, b, and c are unknown indices.

The dimensional equation is

$$T = M^a L^b (L/T^2)^c$$

or

$$T = M^a L^{b+c} T^{-2c}.$$

Equating indices

of M, $a = 0,$

of L, $b + c = 0,$

of T, $-2c = 1.$

From these equations,

$$c = -\tfrac{1}{2} \quad \text{and} \quad b = -c = \tfrac{1}{2},$$

therefore $T = L^{\frac{1}{2}}(L/T^2)^{-\frac{1}{2}}$

and the required equation is

$$t = C(l/g)^{\frac{1}{2}}.$$

Example 3.—To find how the deflection y of a beam varies with the linear dimensions and the modulus of elasticity E, given that y is directly proportional to the applied load W.

Assume $y = CWd^a E^b,$

where C is a constant and d may be any linear dimension of the beam, such as span, depth, breadth, thickness of a flange, etc. Since E = stress/strain, its dimensions are those of stress.

The dimensional equation is

$$L = FL^a (F/L^2)^b$$

or

$$L = F^{1+b} L^{a-2b}.$$

Equating indices

of F, $1 + b = 0,$

of L, $a - 2b = 1.$

From these equations,

$$b = -1 \quad \text{and} \quad a = 1 + 2b = -1.$$

Therefore $y = \dfrac{CW}{dE}.$

59. Dimensional Method continued—Dynamical Similarity.—In the examples in the preceding Art. values were obtained for all the indices, but in many cases this is not possible.

Consider the relation

$$S = Cx^a y^b z^c,$$

where S, x, y, and z are physical quantities, C is a constant, and a, b, and c are indices to be determined. Now the right-hand side might have to be represented by a series of terms instead of by one term. If the values of a, b, and c can be obtained when the right-hand side is taken as being one term, it is evident that this one term could not be expanded as a series because the separate terms would all be identical except for their coefficients and could be added together to make one term. But it is not until the indices have been found that it is known that there is only one term.

If S is a function of x, y, and z, or in symbols

$$S = \phi(x, y, z),$$

where ϕ denotes "a function of" (f or F are often used instead of ϕ), it should be assumed that

$$S = \Sigma Cx^a y^b z^c,$$

where Σ denotes that the sum of a number of terms is intended and C is a constant in each term.

Physical considerations require that all the terms of the series must have the same dimensions as the quantity which the series represents, hence the dimensions of the function itself ($\Sigma Cx^a y^b z^c$) may be equated to those of the quantity it represents (S). In cases where all the indices can be found, the function can be expressed in one term.

When equating indices it saves space to use the notation [M] to denote "equating indices of M," and similarly for [L] and [T] and other dimensions which may occur.

It has been stated above that in many cases it is not possible to obtain the values of all the assumed indices; it should be noted, however, that it may be possible to

determine enough indices to provide some useful information, and this is the basis of the solution of problems in dynamical similarity. In considering dynamical similarity, the idea in general is to study the behaviour of one body and to predict what will be the behaviour of a similar body. For instance, information about the design of full-size ships can be obtained from experiments on models.

The examples which follow will help to make the subject clear. The first is one in which all the indices can be found.

Example 1.—In a grinding wheel of outside radius r it is required to find how the permissible angular velocity ω varies with r, with the stress f in the material of the wheel and with its density ρ. (See *Grinding Machinery* by J. J. Guest, p. 29. Arnold.)

Assume $$\omega = \Sigma a f^l r^m \rho^n,$$

where a is a constant and l, m, and n are indices to be determined.

The dimensional equation is

$$\frac{1}{\mathrm{T}} = \left(\frac{\mathrm{ML}}{\mathrm{T^2L^2}}\right)^l \mathrm{L}^m \left(\frac{\mathrm{M}}{\mathrm{L^3}}\right)^n$$

or $$\mathrm{T^{-1} = M^{l+n}L^{-l+m-3n}T^{-2l}}.$$

[T] $$2l = 1$$

[M] $$l + n = 0$$

[L] $$-l + m - 3n = 0.$$

From these equations,

$$l = \tfrac{1}{2}, \quad n = -\tfrac{1}{2}, \quad \text{and} \quad m = -1.$$

Therefore $$\omega = a f^{\frac{1}{2}} r^{-1} \rho^{-\frac{1}{2}}$$

or $$\omega r = a\sqrt{\frac{f}{\rho}}.$$

Regarding the density as constant, the limiting value of the circumferential speed ωr is proportional to the square root of the permissible stress f.

Example 2.—To find how the deflections y of geometrically similar beams vary with the linear dimensions d, the modulus of elasticity E, and the applied load W.

This problem is similar to Example 3 in the preceding Art., but there it was given that y is directly proportional to W.

Since y is a function of W, d, and E, or

$$y=\phi_1(W, d, E),$$

assume $$y=\Sigma AW^a d^b E^c,$$

where A is a constant in each term and a, b, and c are unknown indices.

The dimensional equation is

$$L=F^a L^b (F/L^2)^c$$

or $$L=F^{a+c}L^{b-2c}.$$

[L] $$b-2c=1$$

[F] $$a+c=0.$$

From these equations, obtaining b and c in terms of a,

$$c=-a \quad \text{and} \quad b=1-2a.$$

As there are three unknown quantities and only two equations, numerical values cannot be obtained for a, b, and c.

Therefore $$y=\Sigma AW^a d^{1-2a}E^{-a}$$

or $$y=d\Sigma A\left(\frac{W}{d^2E}\right)^a$$

or $$\frac{y}{d}=\phi_2\left(\frac{W}{d^2E}\right).$$

Both y/d and W/d^2E are dimensionless. As the index a is unknown, the function of W/d^2E may or may not be a series, but if it is a series each term will be dimensionless and the equation will still be dimensionally correct.

It follows that, in geometrically similar beams, if the values of W/d^2E are equal, then corresponding values of y/d will be equal.

For instance, suppose the linear dimensions (d_1) of one

beam are ten times those (d_2) of another beam, and let W_1 and W_2 respectively be similarly situated loads on the two beams.

If $$\frac{W_1}{d_1^2 E} = \frac{W_2}{d_2^2 E} \quad \text{or} \quad \frac{W_1}{(10d_2)^2 E} = \frac{W_2}{d_2^2 E},$$

then, if E is the same for each beam—that is, if the beams are both made of the same material—W_1 must be 100 times W_2.

Now $$\frac{y_1}{d_1} = \frac{y_2}{d_2} \quad \text{or} \quad \frac{y_1}{10d_2} = \frac{y_2}{d_2},$$

therefore $$y_1 = 10y_2.$$

Example 3.—The resistance to the motion of a ship is

$$R = \phi_1(l, v, \rho, \nu, g),$$

where l is any linear dimension of the ship, v is the speed, ρ is the density of the water and ν is its kinematic viscosity (dimensions L^2/T), and g is the acceleration due to gravity. It is required to compare the resistance of a full-size ship with that of a model.

Assume $$R = \Sigma A l^a v^b \rho^c \nu^d g^e,$$

where A is a constant in each term, and a, b, c, d, and e are unknown indices.

The dimensional equation is

$$\frac{ML}{T^2} = L^a\left(\frac{L}{T}\right)^b\left(\frac{M}{L^3}\right)^c\left(\frac{L^2}{T}\right)^d\left(\frac{L}{T^2}\right)^e$$

or $$MLT^{-2} = L^{a+b-3c+2d+e}M^cT^{-b-d-2e}.$$

[M] $$c = 1$$

[L] $$a + b - 3c + 2d + e = 1$$

[T] $$-b - d - 2e = -2,$$

from which

$$c = 1, \quad b = 2 - d - 2e, \quad \text{and} \quad a = 2 - d + e.$$

Therefore $\quad \mathrm{R} = \Sigma \mathrm{A} l^{2-d+e} v^{2-d-2e} \rho \nu^d g^e$

$$= \Sigma \mathrm{A} \rho l^2 v^2 (l^{-d+e} v^{-d-2e} \nu^d g^e)$$

$$= \rho l^2 v^2 \Sigma \mathrm{A} \left(\frac{\nu}{lv}\right)^d \left(\frac{lg}{v^2}\right)^e$$

$$= \rho l^2 v^2 \phi_2 \left(\frac{\nu}{lv}, \frac{lg}{v^2}\right).$$

The ratio lv/ν is known as *Reynolds' Number* and is dimensionless, as is lg/v^2. Now function $1/x$ is also a function of x and therefore, writing Reynolds' Number instead of its reciprocal,

$$\mathrm{R} = \rho l^2 v^2 \phi_3 \left(\frac{lv}{\nu}, \frac{lg}{v^2}\right)$$

or

$$\frac{\mathrm{R}}{\rho l^2 v^2} = \phi_3 \left(\frac{lv}{\nu}, \frac{lg}{v^2}\right).$$

It should be noted that both sides of the equation are now dimensionless.

If it is arranged that lv/ν and lg/v^2 are each the same for a model as for a ship, then $\mathrm{R}/\rho l^2 v^2$ will be the same for a model as for a ship.

Example 4.—The amplitude of vibration of a rotating body is

$$\delta = \Sigma \mathrm{A} l^r \rho^s \mathrm{E}^t \mathrm{C}^u \mathrm{I}^v \mathrm{J}^x \omega^y g^z,$$

where A is a constant in each term, l is a linear dimension of the body, ρ is the density, E is the modulus of elasticity, C is the modulus of rigidity, I is the moment of inertia of a cross-section about a transverse axis through the centre of gravity of the cross-section, J is the polar moment of inertia of a cross-section, ω is the angular velocity, g is the acceleration due to gravity, and r, s, t, u, v, x, y, and z are unknown indices. It is required to investigate the motions of geometrically similar bodies.

The dimensional equation is

$$\mathrm{L} = \mathrm{L}^r \left(\frac{\mathrm{M}}{\mathrm{L}^3}\right)^s \left(\frac{\mathrm{ML}}{\mathrm{T}^2\mathrm{L}^2}\right)^t \left(\frac{\mathrm{ML}}{\mathrm{T}^2\mathrm{L}^2}\right)^u \mathrm{L}^{4v} \mathrm{L}^{4x} \left(\frac{1}{\mathrm{T}}\right)^y \left(\frac{\mathrm{L}}{\mathrm{T}^2}\right)^z$$

or $\quad L = L^{r-3s-t-u+4v+4x+z} M^{s+t+u} T^{-2t-2u-y-2z}.$

$$[L] \qquad r - 3s - t - u + 4v + 4x + z = 1$$

$$[M] \qquad s + t + u = 0$$

$$[T] \qquad -2t - 2u - y - 2z = 0.$$

From these equations

$$y = -2t - 2u - 2z,$$

$$s = -t - u,$$

and $$r = 1 - 2t - 2u - 4v - 4x - z.$$

Therefore

$$\delta = \Sigma A l^{1-2t-2u-4v-4x-z} \rho^{-t-u} E^t C^u I^v J^x \omega^{-2t-2u-2z} g^z$$

$$= l \Sigma A (l^{-2t} \rho^{-t} E^t \omega^{-2t})(l^{-2u} \rho^{-u} C^u \omega^{-2u})(l^{-4v} I^v)(l^{-4x} J^x)(l^{-z} \omega^{-2z} g^z)$$

$$= l \Sigma A \left(\frac{E}{l^2 \rho \omega^2}\right)^t \left(\frac{C}{l^2 \rho \omega^2}\right)^u \left(\frac{I}{l^4}\right)^v \left(\frac{J}{l^4}\right)^x \left(\frac{g}{l\omega^2}\right)^z$$

$$= l \phi\left(\frac{E}{l^2 \rho \omega^2}, \frac{C}{l^2 \rho \omega^2}, \frac{I}{l^4}, \frac{J}{l^4}, \frac{g}{l\omega^2}\right).$$

Dividing by l, the left-hand side of the equation becomes δ/l which is dimensionless, and it will of course be found that the terms on the right-hand side are also dimensionless.

Suppose transverse vibrations are being considered, then $C/l^2\rho\omega^2$ and J/l^4 will not enter into the problem and I/l^4 will be the same for geometrically similar bodies.

The relationship is now reduced to

$$\frac{\delta}{l} = \phi\left(\frac{E}{l^2 \rho \omega^2}, \frac{g}{l\omega^2}\right).$$

If $g/l\omega^2$ can be neglected and if $E/l^2\rho\omega^2$ is the same for a model as for the actual body, then δ/l will be the same for each. It is not practicable to vary $g/l\omega^2$ and $E/l^2\rho\omega^2$ simultaneously so that each is the same for a model as for the actual body.

The inclusion of the moment of inertia I as one of the variables in the initial equation allows the model to have a differently shaped cross-section from that of the actual body. Taking l to represent longitudinal measurements

only, then provided I/l^4 is the same for the model as for the actual body, the cross-sections need not be similar.

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Exercises VI

Check the dimensions of the equations given in Exercises 1 to 9.

1. The resistance to the penetration of a pile is given by

$$R = W + w + \frac{W^2 h}{(W + w)d}$$

where W and w are weights and h and d are lengths.

2. Power transmitted by a wire rope is given by

$$H = \left(f - \frac{Ed}{D} - \frac{wv^2}{g}\right)\frac{(1-n)v}{550}$$

where H is horse-power transmitted per unit net section of the rope, f is maximum stress, E is modulus of elasticity (a stress), d is diameter of each wire of rope, D is diameter of pulley, w is weight per unit length per unit net section of rope, v is velocity of rope, g is acceleration due to gravity, and $n = e^{-\mu\theta}$ where μ is a constant and θ is an angle.

3. The maximum horse-power transmitted by a belt is

$$H = \frac{bt(1-n)}{550} \times \tfrac{2}{3} f \sqrt{\frac{fg}{3w}}$$

where b and t are the breadth and thickness, n is a numerical fraction, f is stress, g is acceleration due to gravity, and w is the weight per unit length per unit area of cross-section.

4. $$Qa = \frac{1}{a}\frac{dv}{dt}\{I_1 + n^2 I_2\}$$

where Q is force, a is length, v is velocity, t is time, I_1 and I_2 are moments of inertia of masses, and n is a constant.

5. Horse-power transmitted by a water turbine is

$$\text{H.P.} = \frac{Q\omega}{550g}\{v_1 r_1 \cos \alpha_1 + v_2 r_2 \cos \alpha_2\}$$

where Q is weight of water per unit time, ω is angular velocity, g is acceleration due to gravity, v_1 and v_2 are linear velocities, r_1 and r_2 are radii, and α_1 and α_2 are angles.

6. $$\theta - \phi = \omega\sqrt{\frac{I}{n}}\sin\left(\sqrt{\frac{n}{I}}t\right) - \frac{f}{n}.$$

θ and ϕ are angles, ω is an angular velocity, I is the moment of inertia of a mass, n is a twisting moment per radian, f is a frictional moment, and t is time.

7. Two of the formulæ for the critical load P on a strut are

$$P = \frac{\pi^2}{l^2}EI \quad \text{and} \quad P = fA \Big/ \left\{1 + a\left(\frac{l}{k}\right)^2\right\}$$

where l is length, E is modulus of elasticity, I is moment of inertia of the area of a cross-section about a transverse axis, f is stress, A is area, k is radius of gyration, and a is a constant.

8. The periodic time t of the torsional vibration of a shaft fixed at one end and carrying a flywheel at the other end is

$$t = 2\pi\sqrt{\frac{32Il}{C\pi d^4}}$$

where I is moment of inertia of flywheel, l is length, C is modulus of rigidity (a stress), and d is diameter of shaft.

9. The following is a formula which occurred in some research work:

$$V^2 = \frac{\pi}{2}H\frac{d^2t^2}{m(t+l)} \Big/ \left\{1 - \frac{\pi\rho d^2t^2}{8m(t+l)}\right\}$$

where V is velocity, H is a Brinell Number (dimensions same as stress), m is mass, ρ is density, and d, t, and l are linear measurements.

---

10. The frequency of vibration of a stretched wire is given by

$$f = \phi(l, P, m)$$

where l is length, P is stretching force, and m is mass per unit length.

Show that
$$f=\frac{A}{l}\sqrt{\frac{P}{m}}$$
where A is a constant.

11. If the periodic time t of a pendulum varies with mass m, length l, the acceleration g due to gravity, and with the arc of swing s on one side of the mean position, show that
$$t=\left(\frac{l}{g}\right)^{\frac{1}{2}}\phi(a)$$
where $a=s/l$ is the amplitude.

12. (*a*) Supposing that in a fluid flywheel the torque transmitted is given by
$$T_q=\phi_1(N, D, \rho)$$
where N is revolutions per unit time, D is a linear dimension, and ρ is the density of the fluid, show that
$$T_q=kN^2D^5\rho$$
where k is a constant.

(*b*) If the torque is a function of viscosity μ as well as of N, D, and ρ, show that
$$T_q=N^2D^5\rho\phi_2(\mu/ND^2\rho).$$

The torque also depends on the percentage slip, that is on $\frac{N-n}{N}\times 100$, where N and n are the speeds of the two parts of the flywheel respectively, but as this quantity is dimensionless it does not enter into the above calculations.

13. If
$$s=\phi_1(u, t, f),$$
where s is displacement, u is velocity, t is time, and f is acceleration, show that
$$s=ut\phi_2\left(\frac{ft}{u}\right).$$

Given $u=5$, $f=2$, and the tabulated corresponding values of

t	1	2	3	4	5
s	6	14	24	36	50

t and s, plot s/ut against ft/u and show that
$$s/ut=1+\tfrac{1}{2}ft/u \quad \text{or} \quad s=ut+\tfrac{1}{2}ft^2.$$

14. If any deflection y of a whirling shaft is a function of length l, mass m per unit length, modulus of elasticity E, moment of inertia I about a diameter of a cross-section, and angular velocity ω, show that

$$y = l\phi_1\left(\frac{\mathrm{E}}{m\omega^2}, \frac{\mathrm{I}}{l^4}\right)$$

and that this may also be written as

$$y = l\phi_2\left(\frac{\mathrm{EI}}{m\omega^2 l^4}, \frac{\mathrm{I}}{l^4}\right).$$

15. Given that $\qquad \mathrm{R} = \phi_1(\rho, v, l, \nu)$

where R is the resistance to the motion of a body through a fluid of density ρ, v is the velocity, l is length, and ν is kinematic viscosity, show that

$$\mathrm{R} = \rho v^2 l^2 \phi_2\left(\frac{vl}{\nu}\right).$$

16. Prove that the resistance of the air to a projectile can be written as

$$\mathrm{R} = \rho v^2 d^2 \phi\left(\frac{v}{a}, \frac{vd}{\nu}\right)$$

where R is air resistance, ρ is air density, a is velocity of sound in air, v is velocity of projectile of diameter d, and ν is kinematic viscosity.

17. Given that

$$f = \phi_1(l, \rho, \mathrm{E}, \mathrm{C}, \mathrm{I}, \mathrm{J}, \omega, g)$$

where f is frequency and the other symbols are those used in Example 4, Art. 59, show that

$$f = \omega\phi_2(\mathrm{E}/l^2\rho\omega^2, \mathrm{C}/l^2\rho\omega^2, \mathrm{I}/l^4, \mathrm{J}/l^4, g/l\omega^2).$$

CHAPTER VII

SIMPLE HARMONIC MOTION

60. Simple Harmonic Motion.—If a point P (Fig. 88) moves with constant speed in a circular path, then a point p which is the projection of P on a fixed diameter XX′ moves with *simple harmonic motion.* Put in another way, it may be said that uniform circular motion looked at edgeways appears to be simple harmonic motion.

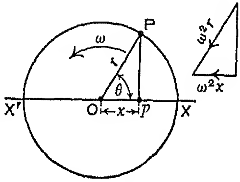

FIG. 88.

Let O be the centre of the circle of radius OP $=r$, let the constant angular velocity of OP be ω, and at time t let the angle XOP be θ and let $Op=x$.

Since P is moving with uniform speed in a circular path, the acceleration of P is $\omega^2 r$ along PO (see Art. 28, p. 47). The component of this acceleration, in the direction parallel to OX, is $-\omega^2 x$ and this is also the acceleration of the point p. The sign is negative because the sense of the acceleration is towards O and the positive direction of x is away from O to the right. When x is positive the acceleration is negative, and when x is negative the acceleration is positive.

The acceleration of the point p is

$$\frac{d^2x}{dt^2} = -\omega^2 x,$$

therefore

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad . \quad . \quad . \quad (1),$$

and this is the equation of simple harmonic motion.

Simple harmonic motion need not be in a straight line; for instance, a flywheel may oscillate with this type of motion which is also approximately that of a simple pendulum. Any body has simple harmonic motion if its acceleration is proportional to its displacement from the mean position and is directed towards the mean position. The equation of motion can always be reduced to the form given in (1).

From Fig. 88 it is seen that

$$x = r \cos \theta \quad . \quad . \quad . \quad (2),$$

and this is a solution of (1), for differentiating with respect to t,

$$\frac{dx}{dt} = -r \sin \theta \frac{d\theta}{dt} \quad \text{and} \quad \frac{d^2x}{dt^2} = -r \cos \theta \left(\frac{d\theta}{dt}\right)^2,$$

then substituting the values of $\frac{d^2x}{dt^2}$ and x in (1), writing ω for the angular velocity $\frac{d\theta}{dt}$,

$$-\omega^2 r \cos \theta + \omega^2 r \cos \theta = 0,$$

showing that (1) is satisfied when $x = r \cos \theta$.

Now suppose θ is measured from a fixed line OA, the angle XOA having a value ϕ (Fig. 89), then as before $\frac{d^2x}{dt^2} = -\omega^2 x$, but in this case

$$x = r \cos (\theta + \phi) \quad . \quad . \quad . \quad (3),$$

and this also is a solution of (1), as may be proved by differentiation and substitution.

Since $\cos (\theta + \phi) = \sin (\theta + \phi + 90°) = \sin (\theta + \phi')$, where $\phi' = \phi + 90°$, it follows that

$$x = r \sin (\theta + \phi') \quad . \quad (4)$$

is another way of expressing the solution given in (3).

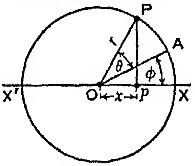

FIG. 89.

Again, since $\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$, therefore, substituting in (3),

$$x=r(\cos\theta\cos\phi-\sin\theta\sin\phi)$$

or
$$x=A\cos\theta+B\sin\theta \quad . \quad . \quad . \quad (5),$$

writing A for $r\cos\phi$ and B for $-r\sin\phi$. Since r, $\cos\phi$, and $\sin\phi$ are constants, therefore A and B are constants. The same result could be obtained by expanding (4).

Equations (3), (4), and (5) are merely different ways of writing the general solution of the differential equation (1), and whichever form of solution is used, the values of the constants are obtained from the conditions given in any particular problem.

Since $\theta=\omega t$, equations (3), (4), and (5) may also be written

$$x=r\cos(\omega t+\phi) \quad . \quad . \quad . \quad (6),$$

$$x=r\sin(\omega t+\phi') \quad . \quad . \quad . \quad (7),$$

$$x=A\cos\omega t+B\sin\omega t \quad . \quad . \quad (8).$$

The velocity is obtained by differentiation; for instance, from (8) the velocity is

$$\frac{dx}{dt}=-\omega A\sin\omega t+\omega B\cos\omega t \quad . \quad . \quad (9).$$

The maximum value of the displacement x is r and is called the *amplitude.* A journey from X to X′ and back to X is called a *cycle*; but a cycle may start at any point and is completed on arrival back at the same point, the motion then being in the same direction as at first. The time taken to complete one cycle is called the *periodic time* or the *period.* The number of cycles completed in unit time is called the *frequency.*

If T is the periodic time and f is the frequency, then, measuring ω in radians per unit time,

$$\omega T=2\pi, \quad T=2\pi/\omega, \quad \text{and} \quad f=1/T=\omega/2\pi.$$

Referring to Fig. 88, the angle XOP is called the *phase angle.* The ratio (*angle* XOP)/2π, which is the same thing

as the ratio (*time taken to travel from* X *to* p)/(*periodic time*), is called the *phase*, but sometimes the term phase is used to denote phase angle. The two simple harmonic motions $x_1 = a \cos(\omega t + \phi_1)$ and $x_2 = b \cos(\omega t + \phi_2)$ differ in phase by the angle $\phi_2 - \phi_1$ or by $(\phi_2 - \phi_1)/2\pi$ of a period and either of these forms is the *phase difference.*

In some problems, instead of using equations (6), (7), or (8), it is convenient to refer back to the corresponding circular motion and work from first principles. The two methods are illustrated in the examples which follow.

Example 1.—A body moves with simple harmonic motion. The amplitude is 3 inches and the periodic time is 5 seconds. It is required to obtain expressions for the displacement, the velocity, and the acceleration in terms of time t.

Since $\omega T = 2\pi$ and $T = 5$ sec., therefore $\omega = 2\pi/5$ rad./sec.

If x is the displacement from the mean position at time t, then the equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

or

$$\frac{d^2x}{dt^2} + \frac{4\pi^2}{25} x = 0.$$

The solution, using the form given in equation (8), is

$$x = A \cos \omega t + B \sin \omega t,$$

and the velocity is

$$\frac{dx}{dt} = -\omega A \sin \omega t + \omega B \cos \omega t.$$

Assuming that time is measured from the instant when the body is at one end of its travel, then $\frac{dx}{dt} = 0$ when $t = 0$, and substituting these values in the velocity equation,

$$0 = 0 + \omega B, \quad \text{or} \quad B = 0.$$

Hence the equation of motion reduces to

$$x = A \cos \omega t.$$

The amplitude is 3 inches, so $x=3$ when $t=0$, therefore $A=3$.

Therefore $$x=3\cos\omega t=3\cos\frac{2\pi}{5}t \text{ inches,}$$

velocity $$\frac{dx}{dt}=-\frac{6\pi}{5}\sin\frac{2\pi}{5}t \text{ in./sec.,}$$

and acceleration $$\frac{d^2x}{dt^2}=-\frac{12\pi^2}{25}\cos\frac{2\pi}{5}t \text{ in./sec.}^2.$$

The same results could have been obtained by using equations (6) or (7).

Example 2.—A ship is rolling with a period of 10 seconds. A man at the masthead 100 feet above the deck is swung to and fro 25 feet on either side of the vertical with a motion which is approximately horizontal and simple harmonic. The man weighs 200 lb., and his horizontal hold failing at 50 lb. he is thrown off the mast. The width of the deck being 80 feet, prove that he falls clear of the ship. [Assume $\pi^2=10$ and $g=32$ f.s. units.] [C.U.]

Since force = mass × acceleration, or $P=Mf$, the acceleration of the man when his hold fails is

$$f=\frac{P}{M}=\frac{50\times 32}{200}=8 \text{ ft./sec.}^2.$$

Now $T=10$ sec., $\omega T=2\pi$, therefore $\omega=\frac{2\pi}{10}=\frac{\pi}{5}$ rad./sec.

The amplitude is 25 ft., therefore the maximum horizontal acceleration of the masthead is

$$25\omega^2=25\left(\frac{\pi}{5}\right)^2=10 \text{ ft./sec.}^2.$$

Since acceleration is proportional to the displacement from the mean position, it follows that the man loses his hold when he is $\frac{8}{10}\times 25$ or 20 ft. from the mean position, and this is approximately $40-20$ or 20 ft. from a vertical line at the side of the ship (Fig. 90), ignoring the slopes of the mast and the deck.

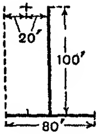

FIG. 90.

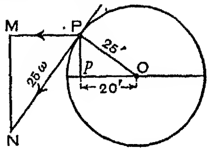

FIG. 91.

The linear velocity of the corresponding circular motion (Fig. 91) is 25ω and, resolving horizontally, the velocity of the man at the instant he loses his hold is $v = 25\omega \frac{PM}{PN}$.

From the similar triangles PMN and PpO, $\frac{PM}{PN} = \frac{Pp}{PO}$.

Therefore

$$v = 25\omega \frac{Pp}{PO} = 25\omega \frac{\sqrt{(25^2 - 20^2)}}{25} = 15\omega = 15 \times \frac{\pi}{5} = 3\pi \text{ ft./sec.}$$

The time taken to travel horizontally another 20 ft. at this velocity is $\frac{20}{3\pi}$ sec.

Let h be the distance fallen in time t, then $h = \frac{1}{2}gt^2$.

When $t = \frac{20}{3\pi}$, $h = \frac{1}{2} \times 32\left(\frac{20}{3\pi}\right)^2 = 71$ ft. approximately.

Therefore the man has fallen 71 ft. by the time he is over the side of the ship, and since he is then $100 - 71 = 29$ ft. above the deck it is evident that he will fall clear of the ship.

61. Simple Pendulum.—A particle A of mass m, suspended from a fixed point O by a light string OA of length l, swings in a vertical plane through a small angle α on each side of the vertical OY (Fig. 92). It is required to find the equation of motion and the periodic time. (The magnitude of the angle α is discussed at the end of Art. 63.)

Suppose that at time t the displacement of the particle A from its mean position Y is s, measured along the arc YA, and that the angle YOA is θ.

Since $s=l\theta$, differentiating twice gives $\frac{d^2s}{dt^2}=l\frac{d^2\theta}{dt^2}$, and this is the acceleration of the particle along the tangent at A, the direction in which s increases being regarded as positive.

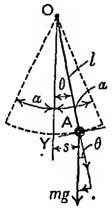

FIG. 92.

The weight of the particle is $w=mg$ and its component along the tangent at A is $mg\sin\theta$ acting in the negative direction.

Now force = mass × acceleration,

therefore
$$-mg\sin\theta=ml\frac{d^2\theta}{dt^2}$$

or
$$\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin\theta=0.$$

When θ is small, $\sin\theta=\theta$ approximately, then

$$\frac{d^2\theta}{dt^2}+\frac{g}{l}\theta=0,$$

and this is the equation of simple harmonic motion.

By comparison with the standard form

$$\frac{d^2\theta}{dt^2}+\omega^2\theta=0,$$

it is seen that $\omega^2=g/l$ or $\omega=\sqrt{(g/l)}$.

Therefore, provided the amplitude a is small, the periodic time

$$T=2\pi/\omega=2\pi\sqrt{(l/g)}.$$

62. **Bifilar Suspension.**—A plate weighing 2W lb. is suspended by two vertical strings a distance $2a$ apart and each of length l (Fig. 93). The centre of gravity is midway between the points of attachment of the strings and in the same horizontal line. The plate is turned through a small angle about the vertical axis through the centre

of gravity and is then released. It is required to investigate the subsequent motion.

Taking time and angular displacement as zero when the position of statical equilibrium is reached, suppose that at time t the plate has turned through an angle θ, the force in each string is F, and that each string then makes an angle ϕ with the vertical.

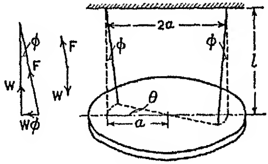

FIG. 93.

Since θ and ϕ are small, $l\phi = a\theta$ or $\phi = a\theta/l$ approximately. Neglecting vertical acceleration, the vertical component of the force F in each string is equal to W, half the weight of the plate, and the horizontal component is approximately $W\phi$, which produces a torque on the plate equal to $W\phi a$.

If T_g is the torque on the plate at time t, then

$$T_g = 2W\phi a = 2Wa^2\theta/l.$$

If $I = \frac{2W}{g}k^2$ is the moment of inertia of the plate about the axis of rotation, then since the torque is negative when θ is positive

$$T_g = -I\frac{d^2\theta}{dt^2} = -\frac{2W}{g}k^2\frac{d^2\theta}{dt^2}.$$

Therefore $$\frac{2Wa^2\theta}{l} = -\frac{2W}{g}k^2\frac{d^2\theta}{dt^2}$$

or $$\frac{d^2\theta}{dt^2} + \frac{ga^2}{lk^2}\theta = 0$$

which is the equation of simple harmonic motion.

By comparison with the standard form

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0,$$

it is seen that $$\omega = \sqrt{\left(\frac{ga^2}{lk^2}\right)}$$

and periodic time

$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\left(\frac{lk^2}{ga^2}\right)}=2\pi\frac{k}{a}\sqrt{\frac{l}{g}}.$$

63. Simple Pendulum—A Closer Approximation to the Value of the Periodic Time.—The equation of motion for a simple pendulum is, from Art. 61,

$$\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin\theta=0,$$

which is true for all values of θ. The solution of this equation involves an elliptic integral which cannot be evaluated exactly and the answer is obtained in the form of a series, the degree of approximation depending on the number of terms used.

Only the main steps of the working are indicated, and those readers who are not interested in the mathematics may pass on to the final result.

Multiplying the equation by $2\frac{d\theta}{dt}$ and integrating, taking the amplitude as α, then

$$\frac{d\theta}{dt}=\sqrt{\frac{2g}{l}}(\cos\theta-\cos\alpha)^{\frac{1}{2}}.$$

Since $\cos\theta=1-2\sin^2\frac{\theta}{2}$ and $\cos\alpha=1-2\sin^2\frac{\alpha}{2}$,

$$\frac{d\theta}{dt}=\sqrt{\frac{2g}{l}}\left(2\sin^2\frac{\alpha}{2}-2\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}.$$

Integrating, $$\int_0^{\frac{1}{4}T}dt=\frac{1}{2}\sqrt{\frac{l}{g}}\int_0^{\alpha}\frac{d\theta}{\left(\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}}$$

where T is the periodic time.

Putting $\sin\frac{\theta}{2}=b\sin\phi$, where $b=\sin\frac{\alpha}{2}$, it can be shown that

$$T=4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}(1-b^2\sin^2\phi)^{-\frac{1}{2}}d\phi.$$

Expanding by the binomial theorem and integrating term by term gives

$$T = 2\pi\sqrt{\frac{l}{g}}\left\{1 + \frac{1}{2^2}b^2 + \frac{3^2}{2^2.4^2}b^4 + \frac{3^2.5^2}{2^2.4^2.6^2}b^6 + \frac{3^2.5^2.7^2}{2^2.4^2.6^2.8^2}b^8 + \ldots\right\}$$

where, as already stated, $b = \sin\frac{a}{2}$ and a is the amplitude. It can be seen now that the periodic time T is not independent of the amplitude a.

Denoting the expression in brackets by F, then

$$T = 2\pi\sqrt{\frac{l}{g}} \times F,$$

and F, being a function of the amplitude a, may be called the *amplitude factor*.

When $a = 10°$,

$$F = 1 + 0{\cdot}001899 + 0{\cdot}000008 + 0{\cdot}000000 = 1{\cdot}001907,$$

therefore $$T = 2\pi\sqrt{\frac{l}{g}} \times 1{\cdot}001907,$$

and the error is less than $\frac{1}{5}$ of 1 per cent. when the factor F is omitted. Evidently an amplitude of 10° might fairly be described as small when the periodic time is being calculated, and in this case the theory of Art. 61 is sufficiently accurate.

The graph of F plotted against a is shown in Fig. 94. When $F = 1{\cdot}01$, the value of a is about 23°, so for this angle the error in the simple formula $T = 2\pi\sqrt{(l/g)}$ is about 1 per cent. When $a = 60°$, the error is about 7 per cent.

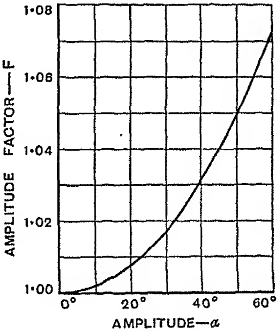

FIG. 94.

Exercises VII

Take $g = 32{\cdot}2$ ft./sec.^{2}

1. In a certain problem the equation of motion is

$$1{\cdot}24\frac{d^2\theta}{dt^2} + 458\theta = 0,$$

the unit of time being the second. Show that the motion is simple harmonic, by comparing the given equation with the standard equation, then find the periodic time and the frequency.

2. A body moving with simple harmonic motion has a frequency of 141 cycles per minute. Using foot and second units, find expressions for the displacement x to satisfy the given conditions in the following cases:—

(i) When $t = 0$, $x = \frac{1}{8}$ ft., and $\frac{dx}{dt} = 0$.

(ii) When $t = 0$, $x = 0$; when $t = \frac{15}{141}$ sec., $x = \frac{1}{8}$ ft.

(iii) When $t = 0$, $x = \frac{1}{16}$ ft., and $\frac{dx}{dt} = -\frac{141\pi}{160\sqrt{3}}$ ft./sec.

3. If in simple harmonic motion the maximum velocity is V, the amplitude is a, and the velocity is v when the displacement from the mean position is x, show that $v = \pm V\sqrt{(1 - x^2/a^2)}$. If $v = 4$ ft./sec. when $x = 1$ ft., and $v = 2$ ft./sec. when $x = 1{\cdot}5$ ft., find the values of a, V, and the periodic time T.

4. A body moving with simple harmonic motion has a maximum velocity of 25 ft./sec. Find its velocity (i) when it is a distance equal to half the amplitude from its mean position, (ii) when it has travelled from its mean position for half the time it takes to reach an extreme position.

5. A body moving in a straight line with simple harmonic motion has a maximum acceleration of 10 ft./sec.^{2} and performs 100 complete oscillations a minute. Find the amplitude and the maximum velocity.

6. The reciprocating parts of a single cylinder engine have a mass weighing 11·4 lb. and the length of the crank is 4·25 inches. Neglecting the obliquity of the connecting-rod and taking the engine speed as 2400 r.p.m., find the maximum velocity and the maximum acceleration of the piston; also find the maximum accelerating force.

7. A body weighing 10 pounds is suspended by a helical spring which is fixed at its upper end (Fig. 95). The spring is 6 inches long when unloaded and 8 inches long when carrying the load. The body is pulled down a distance of 0·75 inch and released. Find the periodic time of the vibrations and the maximum velocity of the body, assuming that there is no air resistance and neglecting the mass of the spring.

FIG. 95.

8. A body is moving harmonically and makes 20 oscillations a second. Find its greatest velocity if the distance between its extreme positions is 4 inches. Determine its velocity and acceleration when 1 inch from the mid-point of its motion.

A flat plate with bodies resting upon it begins to oscillate vertically through a distance 4 inches: determine within what limit the number of vibrations per minute must be kept if the bodies upon the plate are not to be thrown off by the vibration. [C.U.]

9. If a point moves with S.H.M., show that the graph connecting its velocity with distance along its path may be represented by a circle, if a particular scale is chosen for the velocity ordinate.

Show also that, during motion from one end of the travel to the other, the mean velocity with respect to the distance is $\frac{\pi}{4}$ times the maximum velocity, and with respect to the time is $\frac{2}{\pi}$ times the maximum velocity. [C.U.]

10. Two axes Ox and Oy are respectively horizontal and vertical. On Ox a point P moves with simple harmonic motion (S.H.M.), the mean position being O. A point Q similarly describes S.H.M. on the axis Oy. Each point makes a complete oscillation in one second, and the maximum displacement from O in each case is 1 foot. The motions are so timed that when Q is at O (moving downwards), P is at the maximum distance to the right from O, *i.e.* 1 foot. Show that, relative to Q, P describes a circle with uniform angular velocity. Show also that the acceleration of P with respect to Q is constant in magnitude (but not in direction), and give its numerical value. [C.U.]

11. For a simple pendulum having a small amplitude the periodic time is approximately $T=2\pi\sqrt{(l/g)}$. Taking logarithms and differentiating, show that

$$\frac{\delta T}{T}=\frac{\delta l}{2l}-\frac{\delta g}{2g} \text{ approximately,}$$

where δT, δl, and δg are small increments in T, l, and g respectively.

Find the periodic time of a simple pendulum whose length is

36 inches, and then find the increase in the periodic time due to an increase of $\frac{1}{2}$ inch in the length.

12. Calculate the length of a clock pendulum whose periodic time is 2 seconds. If the length is increased by 0·2 per cent., find the number of seconds a day the clock would gain or lose.

13. A homogeneous disc of uniform thickness and 6 inches in diameter is suspended with its axis vertical by two vertical light threads 2 inches apart. The threads are 2 feet long and are symmetrically situated with regard to the centre of the disc so that each bears half the weight.

The disc is turned about its axis through a small angle and then released. Show that the resulting motion is very nearly simple harmonic, and find the time of a complete oscillation.

If the disc is replaced by one of twice its diameter, find the new time of oscillation. [C.U.]

14. Find the value of the amplitude factor F, correct to four places of decimals, in the formula $T = 2\pi(l/g)^{\frac{1}{2}}F$ when the amplitude is 30°.

CHAPTER VIII

ANALYSIS OF CAMS

64. Internal-Combustion Engine Cams.—The cams discussed in this chapter are of the types used for operating the valves in internal-combustion engines, where the cams rotate and give reciprocating motion to the tappets and valves. A small clearance between a valve and its tappet ensures that the former closes properly on to its seat when the engine is hot, but when the tappet has moved through a distance equal to the clearance and touches the valve stem, then the two pieces move together and continue to do so until the valve closes again. In general the tappet and valve will be briefly described as the *follower.*

Mathematical expressions are found for the displacement, velocity, and acceleration of the follower, the velocity and acceleration being obtained by differentiation. It is important to have a knowledge of the acceleration in order that a suitable valve spring may be designed to deal with the inertia forces.

Problems on cams which are used for various purposes are included in the exercises at the end of this chapter. A vector method is given in Ex. 8, p. 131, and this should be studied carefully.

65. Types of Cams.—The cams which will be analysed are shown in Figs. 96 to 99. Each cam rotates about a centre O with a constant angular velocity, and a follower F reciprocates along an axis passing through the centre O. In Fig. 96 the follower has a flat foot which bears on the cam, but in each of the other three types the contact is between the cam and a roller which is attached to the follower. The profiles of the cams are made up of circular arcs except in Fig. 97, where AB and A′B′ are straight

lines. The cams will be called *straight*, *convex*, or *concave*, according as the lines AB and A′B′ are straight, convex, or concave.

66. The Cycle of Operations.—The motion of the follower F (Figs. 96 to 99) begins when the contact between it and the cam is at A. As the cam turns and the contact moves from A to B, the cam accelerates the follower, and the velocity of the latter reaches a maximum when the contact is at B. From B to D the follower is retarded by a spring,

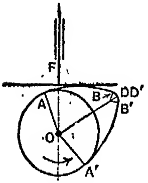

Fig. 96.

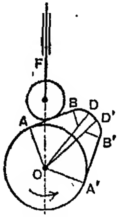

Fig. 97.

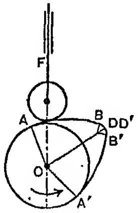

Fig. 98.

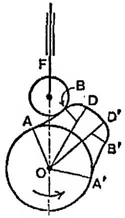

Fig. 99.

not shown, so that the velocity is zero when the contact is at D and the follower has its full lift. The arc DD′ (Figs. 97 and 99) has its centre at O, the centre of rotation, therefore the follower is at rest whilst the contact is on this arc. The period of rest is sometimes called a *dwell*. In Figs. 96 and 98 the points D and D′ coincide and so there is no dwell at full lift.

From D′ to B′ the follower is accelerated downwards by the spring and has its maximum velocity at B′. Finally the cam retards the follower from B′ to A′, and at A′ the follower is again at rest. From A′ to A the follower remains at rest and then the cycle begins again at A.

The cams shown are symmetrical, and therefore in each case it will only be necessary to examine the motion of the follower whilst the valve is opening. Any modification of the dimensions on the closing side of the cam would, of course, require investigation in practice, but the same set of equations would be used with different constants.

In the Articles which follow, AB and BD will be described as the *first curve* and *second curve* respectively.

67. Convex Cam with Flat-Footed Follower.—The cam is shown in two positions in Figs. 100 and 101. The centres of the first curve AB and the second curve BD are at c_1 and c_2 respectively, and the various radii are as indicated. The length OD, although not a radius in this case, is marked r_5 because in two other examples the cams are drawn with a dwell at full lift. To ensure smooth

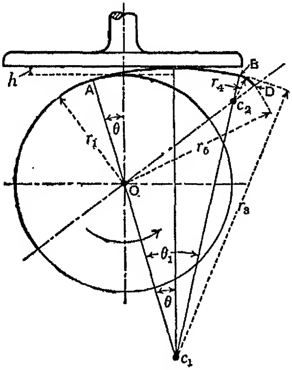

FIG. 100.

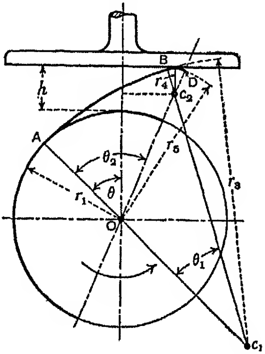

FIG. 101.

running, any two connected arcs should be drawn so as to have a common tangent at the junction.

First Curve AB (Fig. 100). Let h be the lift of the follower when the cam has turned through an angle θ, taking θ as zero when the contact is at A. On account of tappet clearance, the valve lift will be less than the tappet lift by an amount equal to the clearance. Let the angle Ac_1B be θ_1.

$$\text{Lift} \qquad h = r_3 - (r_3 - r_1)\cos\theta - r_1$$
$$= (r_3 - r_1)(1 - \cos\theta).$$

$$\text{Velocity} \qquad v=\frac{dh}{dt}=\frac{dh}{d\theta}\frac{d\theta}{dt}$$

$$=\frac{d\theta}{dt}(r_3-r_1)\sin\theta.$$

$$\text{Acceleration } \frac{dv}{dt}=\frac{dv}{d\theta}\frac{d\theta}{dt}$$

$$=\left(\frac{d\theta}{dt}\right)^2(r_3-r_1)\cos\theta.$$

These three equations hold from $\theta=0$ to $\theta=\theta_1$. The acceleration will have its greatest value when $\cos\theta$ is a maximum—that is, when $\theta=0$.

Second Curve BD (Fig. 101). Let the angle AOD be θ_2.

$$\text{Lift} \qquad h=(r_5-r_4)\cos(\theta_2-\theta)+r_4-r_1.$$

$$\text{Velocity} \qquad v=\frac{dh}{dt}=\frac{dh}{d\theta}\frac{d\theta}{dt}$$

$$=\frac{d\theta}{dt}(r_5-r_4)\sin(\theta_2-\theta).$$

$$\text{Acceleration } \frac{dv}{dt}=\frac{dv}{d\theta}\frac{d\theta}{dt}$$

$$=-\left(\frac{d\theta}{dt}\right)^2(r_5-r_4)\cos(\theta_2-\theta).$$

These three equations hold from $\theta=\theta_1$ to $\theta=\theta_2$. The acceleration will have its greatest value when $\cos(\theta_2-\theta)$ is a maximum—that is, when $\theta=\theta_2$.

To keep the follower in contact with the cam whilst the second curve is in operation, a spring is required which will give to the follower a downward acceleration which is not less than

$$\left(\frac{d\theta}{dt}\right)^2(r_5-r_4) \quad \text{when} \quad \theta=\theta_2,$$

and not less than

$$\left(\frac{d\theta}{dt}\right)^2(r_5-r_4)\cos(\theta_2-\theta_1) \quad \text{when} \quad \theta=\theta_1.$$

The total lift is $r_5 - r_1$, and the value of the lift when $\theta = \theta_1$ may be calculated from either of the lift equations.

Units.—If the uniform angular velocity of the cam, $\frac{d\theta}{dt}$, is in radians per second and the radii are in feet, then the lift h will be in feet, the velocity will be in feet per second, and the acceleration will be in feet per second per second.

68. Spring Force.—Given the acceleration and the mass, it is required to find the force. If the spring force is P when the acceleration is f, and if M is the mass of the follower, including the spring washer, etc., and one-third the mass of the spring, then the spring force may be calculated from the formula

$$P = Mf.$$

69. Straight Cam with Roller Follower.—In Figs. 102 and 103 the centre s of the roller moves along the straight line OF; also, relative to the cam, the point s moves along the straight line A_1B_1 and the circular arc B_1D_1. Taking the radius of the roller as r_2, then A_1B_1 is parallel to AB and at a perpendicular distance r_2 from AB, and the radius of the arc B_1D_1 is $r_2 + r_4$ where r_4 is the radius of the arc BD, both arcs being drawn with centre C_2. If the follower made point contact or knife-edge contact at s with the cam $A_1B_1D_1$, then $A_1B_1D_1$ is the equivalent cam which would give the same motion to the knife-edge follower as is given to the roller follower by the cam ABD.

First Curve AB or A_1B_1 (Fig. 102). Let h be the lift when the cam has turned through an angle θ and let the angle A_1OB_1 be θ_1.

$$\text{Lift} \qquad h = \frac{r_1 + r_2}{\cos\theta} - (r_1 + r_2).$$

$$\text{Velocity} \qquad v = \frac{dh}{dt} = \frac{dh}{d\theta}\frac{d\theta}{dt}$$

$$= \frac{d\theta}{dt}(r_1 + r_2)\frac{\tan\theta}{\cos\theta}.$$

$$\text{Acceleration } \frac{dv}{dt}=\frac{dv}{d\theta}\frac{d\theta}{dt}$$

$$=\left(\frac{d\theta}{dt}\right)^2(r_1+r_2)\frac{1+2\tan^2\theta}{\cos\theta}.$$

These equations hold from $\theta=0$ to $\theta=\theta_1$. As θ increases, $\tan\theta$ increases and $\cos\theta$ decreases, therefore the acceleration increases and its value will be greatest when $\theta=\theta_1$.

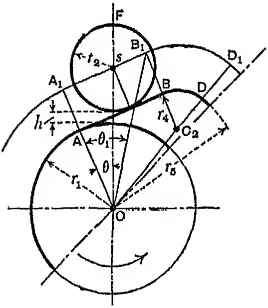

Fig. 102.

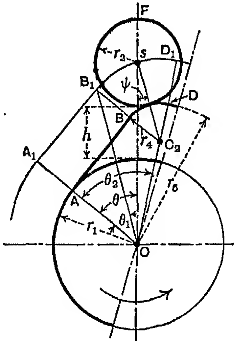

Fig. 103.

Second Curve BD or B_1D_1 (Fig. 103). The cam is shown with a dwell at full lift, but this does not affect the analysis. Let the angle A_1OD_1 be θ_2 and let the angle OsC_2 be ψ, a variable; also, to shorten the work which follows, let $r_5-r_4=d$ and $r_2+r_4=e$.

From the Figure, $(r_2+r_4)\sin\psi=(r_5-r_4)\sin(\theta_2-\theta)$

or $$e\sin\psi=d\sin(\theta_2-\theta),$$

therefore $$\sin\psi=\frac{d}{e}\sin(\theta_2-\theta)$$

and $$\cos\psi=(1-\sin^2\psi)^{\frac{1}{2}}$$

$$=\frac{1}{e}\{e^2-d^2\sin^2(\theta_2-\theta)\}^{\frac{1}{2}}.$$

Lift $\quad h = d\cos(\theta_2 - \theta) + e\cos\psi - (r_1 + r_2)$

$$= d\cos(\theta_2 - \theta) + \{e^2 - d^2\sin^2(\theta_2 - \theta)\}^{\frac{1}{2}} - (r_1 + r_2).$$

Velocity $v = \dfrac{dh}{dt} = \dfrac{dh}{d\theta}\dfrac{d\theta}{dt}$

$$= \frac{d\theta}{dt}\left[d\sin(\theta_2 - \theta) + \frac{d^2\sin 2(\theta_2 - \theta)}{2\{e^2 - d^2\sin^2(\theta_2 - \theta)\}^{\frac{1}{2}}}\right].$$

Acceleration $\dfrac{dv}{dt} = \dfrac{dv}{d\theta}\dfrac{d\theta}{dt}$

$$= \left(\frac{d\theta}{dt}\right)^2\left[-d\cos(\theta_2 - \theta) - \frac{d^2\cos 2(\theta_2 - \theta)}{\{e^2 - d^2\sin^2(\theta_2 - \theta)\}^{\frac{1}{2}}} - \frac{d^4\sin^2 2(\theta_2 - \theta)}{4\{e^2 - d^2\sin^2(\theta_2 - \theta)\}^{\frac{3}{2}}}\right]$$

which after simplification

$$= -\left(\frac{d\theta}{dt}\right)^2\left[d\cos(\theta_2 - \theta) + \frac{d^2\{e^2\cos 2(\theta_2 - \theta) + d^2\sin^4(\theta_2 - \theta)\}}{\{e^2 - d^2\sin^2(\theta_2 - \theta)\}^{\frac{3}{2}}}\right].$$

These equations hold from $\theta = \theta_1$ to $\theta = \theta_2$. Whether the acceleration has its greatest value when $\theta = \theta_1$ or when $\theta = \theta_2$ depends on the values of d and e. The graphs in Fig. 108, Art. 72, illustrate this fact.

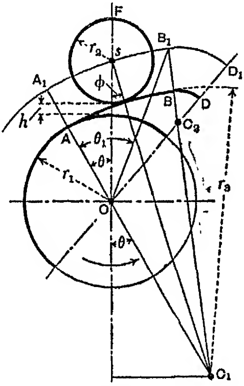

Fig. 104.

70. Convex Cam with Roller Follower.—In this case the first curve AB is a circular arc with centre C_1 and of radius r_3 (Fig. 104). The equivalent cam is $A_1B_1D_1$ as described in the preceding Art., except that here the part A_1B_1 is a circular arc with centre C_1 and of radius $r_3 + r_2$.

First Curve AB or A_1B_1. Let h be the lift when the cam has turned through an

angle θ, let the angle A_1OB_1 be θ_1, and let the angle OsC_1 be ϕ, a variable; also let $r_3 - r_1 = a$ and $r_3 + r_2 = b$.

From the Figure, $(r_3 + r_2) \sin \phi = (r_3 - r_1) \sin \theta$,

or $$b \sin \phi = a \sin \theta,$$

therefore $$\sin \phi = \frac{a}{b} \sin \theta$$

and $$\cos \phi = (1 - \sin^2 \phi)^{\frac{1}{2}} = \frac{1}{b}(b^2 - a^2 \sin^2 \theta)^{\frac{1}{2}}.$$

Lift $$h = b \cos \phi - a \cos \theta - (r_1 + r_2)$$
$$= -a \cos \theta + \{b^2 - a^2 \sin^2 \theta\}^{\frac{1}{2}} - (r_1 + r_2).$$

Velocity $$v = \frac{dh}{dt} = \frac{dh}{d\theta}\frac{d\theta}{dt}$$
$$= \frac{d\theta}{dt}\left[a \sin \theta - \frac{a^2 \sin 2\theta}{2(b^2 - a^2 \sin^2 \theta)^{\frac{1}{2}}}\right].$$

Acceleration $$\frac{dv}{dt} = \frac{dv}{d\theta}\frac{d\theta}{dt}$$
$$= \left(\frac{d\theta}{dt}\right)^2\left[a \cos \theta - \frac{a^2 \cos 2\theta}{(b^2 - a^2 \sin^2 \theta)^{\frac{1}{2}}} - \frac{a^4 \sin^2 2\theta}{4(b^2 - a^2 \sin^2 \theta)^{\frac{3}{2}}}\right]$$

which after simplification
$$= \left(\frac{d\theta}{dt}\right)^2\left[a \cos \theta - \frac{a^2(b^2 \cos 2\theta + a^2 \sin^4 \theta)}{(b^2 - a^2 \sin^2 \theta)^{\frac{3}{2}}}\right].$$

These equations hold from $\theta = 0$ to $\theta = \theta_1$.

Second Curve BD or B_1D_1. The equations are the same as those obtained for the second curve of the straight cam with the roller follower in the preceding Art.

71. Concave Cam with Roller Follower.—The concave cam (Fig. 105) is the same as the convex cam (Fig. 104) except for the change of position of C_1, the centre of the first curve AB. Therefore the formulæ derived for the first curve of the convex cam with roller follower (Art. 70) may be applied to the concave cam by changing the sign of the radius r_3. However, for the sake of clearness, the analysis will be outlined for the concave cam.

First Curve AB or A_1B_1. Let the angle A_1OB_1 be θ_1 and let the angle FsC_1 be ϕ, a variable; also let $r_3+r_1=a$ and $r_3-r_2=b$.

Now $$(r_3-r_2)\sin\phi=(r_3+r_1)\sin\theta$$

or $$b\sin\phi=a\sin\theta,$$

therefore $$\cos\phi=\frac{1}{b}\{b^2-a^2\sin^2\theta\}^{\frac{1}{2}}.$$

Lift $$\begin{aligned}h&=a\cos\theta-b\cos\phi-(r_1+r_2)\\&=a\cos\theta-\{b^2-a^2\sin^2\theta\}^{\frac{1}{2}}-(r_1+r_2).\end{aligned}$$

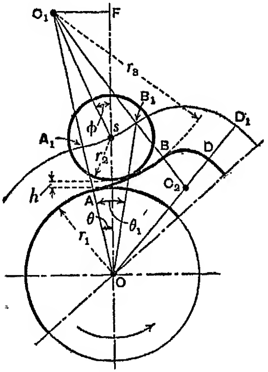

FIG. 105.

These equations should be compared with the corresponding equations for the convex cam with roller follower. (If r_3 had been made negative in the convex cam equations, then $-r_3-r_1=a$ or $r_3+r_1=-a$, and $-r_3+r_2=b$ or $r_3-r_2=-b$. With these changes, the convex cam lift equations give the concave cam lift.)

Since there are only two alterations in sign, the velocity and acceleration equations will be as follows:—

Velocity $$v=\frac{d\theta}{dt}\left[-a\sin\theta+\frac{a^2\sin 2\theta}{2(b^2-a^2\sin^2\theta)^{\frac{1}{2}}}\right].$$

$$\text{Acceleration } \frac{dv}{dt}=\left(\frac{d\theta}{dt}\right)^2\left[-a\cos\theta+\frac{a^2(b^2\cos 2\theta+a^2\sin^4\theta)}{(b^2-a^2\sin^2\theta)^{\frac{3}{2}}}\right].$$

As before, the three equations hold from $\theta=0$ to $\theta=\theta_1$.

Second Curve BD or B_1D_1. The equations are the same as those obtained for the second curve of the straight cam with roller follower (Art. 69).

72. Graphs of Lift, Velocity, and Acceleration.—The graphs * shown in Figs. 106 to 108 refer to two concave

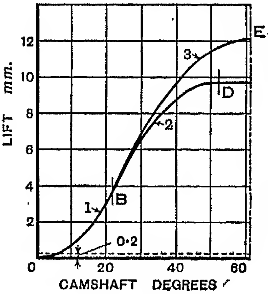

Fig. 106.

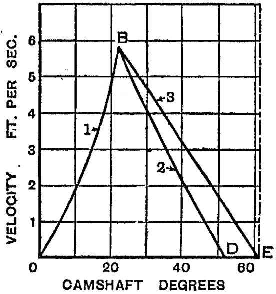

Fig. 107.

cams having roller followers. The curves labelled 1 and 2 in each of the three Figures concern a concave cam and roller having the following dimensions: $r_1=14{\cdot}75$ mm., $r_2=7{\cdot}5$ mm., $r_3=35{\cdot}0$ mm., $r_4=6{\cdot}0$ mm., and $r_5=24{\cdot}5$ mm. The crankshaft speed is 1400 r.p.m., and so the camshaft speed is 700 r.p.m. The maximum tappet lift is $r_5-r_1=24{\cdot}5-14{\cdot}75=9{\cdot}75$ mm. The lift, velocity,

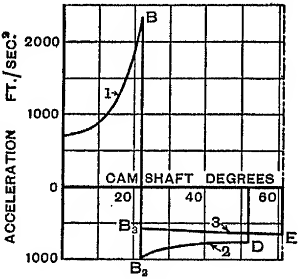

Fig. 108.

* From an article, "Internal-Combustion Engine Cams," by B. B. Low, *Engineering*, May 25, 1923.

and acceleration are given by the curves labelled 1 when the first curve of the cam is in operation, and by the curves labelled 2 when the second curve of the cam is in operation. The other cam and the curves 3 will be mentioned in the next Art.

The clearance between the tappet and the valve stem is 0·2 mm., or 0·0079 inch approximately. The clearance angle, or the angle through which the cam turns whilst the clearance is being taken up, is found by calculation from the lift equation to be $5° \ 41'$. The question of clearance is important because the cam should be designed so that it begins to open the valve at the right moment.

At B the valve has its maximum velocity and there is a sudden change from positive to negative acceleration. At D the valve is fully open, its velocity is zero, and its acceleration is as shown. This particular cam was designed to turn through an angle of approximately 111° whilst the valve is in operation. Therefore half the cam angle is $5° \ 41' + 55° \ 30' = 61° \ 11'$. By calculation, using the given dimensions of the cam, $\theta_1 = 22° \ 6'$ (the angle at B on the curves) and $\theta_2 = 51° \ 53'$ (the angle at D on the curves).

Since θ_2 is less than half the cam angle, there is a dwell at full lift. The angle turned through by the cam during the dwell is $2(61° \ 11' - 51° \ 53') = 18° \ 36'$.

A valve spring is needed to provide the moving mass with the negative acceleration shown at the point B_2. The spring will provide more than enough acceleration at the point D, but this cannot be avoided. The spring force is referred to in Ex. 12, p. 131.

73. Elimination of Dwell by Increasing Lift.—It is shown in Fig. 109 how a dwell at full lift may be eliminated by increasing the lift. Before the design is altered, the second curve is BD with its centre at C_2 and its radius is C_2B. The lift is complete at D and the dwell begins. To make the alteration, let BC_2 be produced to intersect the centre line OE at C_3, then with centre C_3 and radius C_3B draw the arc BE, as shown dotted in the figure. With BE as the second curve of the cam, the lift is complete at E, on the centre line, and there is no dwell.

Before examining the effects of the alteration, the length C_3B must be known. For the cam having the dimensions given in the preceding Art., $C_2B = r_4 = 6$ mm. Half the cam angle is the angle AOE, which is 61° 11′. The length C_2C_3 is approximately 3 mm., which will be taken as the exact value, then $C_3B = 9$ mm., and by calculation the angle AOE is found to be 61° 5′, which is near enough to its given value.

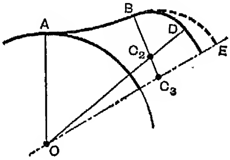

Fig. 109.

By calculation the maximum lift, OE − OA, is equal to 12·1 mm. With the original cam the maximum lift was 9·75 mm., therefore the increase in lift is $12{\cdot}1 - 9{\cdot}75 = 2{\cdot}35$ mm. Allowing for clearance, the increase in lift is

$$\frac{2{\cdot}35}{9{\cdot}75 - 0{\cdot}2} \times 100 = 24{\cdot}6 \text{ per cent.}$$

Now examine Figs. 106 to 108 and compare the curves labelled 2 and 3. The curves 3 refer to the cam with the increased lift. The point which the author wishes to emphasise is that the increased lift actually gives a reduction in the retardation. In foot and second units the value at B_2 is 990, but at B_3 it is 584 and at E it is 655. Therefore the spring force required at the beginning of the second curve (that is, at the point B on the actual cam in Fig. 109) is reduced to $\frac{584}{990}$, or 59 per cent., of its original value and this is certainly an advantage.

Increasing the lift of a valve might cause it to foul the cylinder-head or the piston; also the bearings of the camshaft would have to be slightly larger in diameter to clear the cams if the camshaft is assembled by being pushed through the bearings from one end. Whenever it is possible, however, it is evident that dwell should be eliminated by increasing the radius of the second curve of the cam, because the increased radius reduces the retardation to be provided by the spring and enables a weaker

spring to be used. Another point is that an increase in lift should make it possible to get more mixture (burnt or unburnt) through the valve.

In conclusion it may be remarked that the reader who wishes to find further information concerning the relative advantages of the various cams may refer to the article already quoted in the footnote on p. 126.

Exercises VIII

1. A cam with a flat-footed follower has the dimensions shown in Fig. 110 which is not drawn to scale. The curves AB and BD are circular arcs. Find the radius r of the curve AB and also the angle θ_1 through which the cam turns whilst this curve is in operation. Assuming that the cam rotates at 1000 r.p.m., find the accelerations of the follower when contact is at the points A, B, and D.

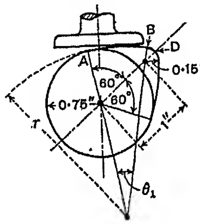

FIG. 110.

2. In Fig. 110 the contact between the cam and the follower begins at the point A and moves to the right, assuming that the cam rotates in an anticlockwise direction. Prove that the contact reaches its extreme right-hand position at the point B.

To reduce uneven wear on the under surface of the flat foot of the follower, this foot is often circular, and the follower is made to rotate about its vertical axis by being offset from the central plane of the cam.

If the vertical axis of the follower passes through the axis of the camshaft and is $\frac{3}{16}$ inch from the central plane of the cam, calculate the greatest distance from the vertical axis of the follower at which wear can occur on the foot, taking the thickness of the cam as $\frac{3}{8}$ inch, the radius $r=4$ inches, and the angle $\theta_1=13°$.

3. A valve of a four-stroke engine is open whilst the crankshaft turns through 220° and is fully open during 28°. The times of opening and closing are equal, and the crankshaft speed is 1800 r.p.m. The lift of the valve (neglecting clearance) is $\frac{3}{8}$ inch, and the acceleration and deceleration are uniform.

(a) Find the value of the acceleration if the valve has its maximum velocity at half lift.

(b) Calculate the lifts corresponding to the camshaft angles 6°, 12°, 18°, 24°, 30°, 36°, and 42°.

(c) Plot curves of acceleration and lift on camshaft angle bases, from 0° to 110°.

4. The profile of a cam is a circle of 3 inches diameter, and the centre about which it rotates is $\frac{1}{2}$ inch from the centre of the circular profile. The follower is provided with a flat palm, at right angles to the line of stroke, which bears directly on the cam surface. The line of stroke is horizontal and passes through the centre of rotation. The weight of the parts actuated by the cam is 5 lb., and a spring is fitted to maintain contact between the cam and the follower. The spring exerts a force of 8 lb. at the beginning and 20 lb. at the end of the out-stroke.

(a) Obtain an expression for the acceleration of the follower in terms of the cam angle.

(b) Find the greatest speed at which the arrangement will run satisfactorily. [U.L.]

5. Find the greatest speed at which the mechanism in the preceding exercise will run satisfactorily if the line of stroke is vertical: (a) when the follower is above the cam; (b) when the follower is below the cam.

6. A fuel injection pump of variable stroke is operated by a compound cam controlled by a governor. Reduced to its essentials, the mechanism is as shown in Fig. 111.

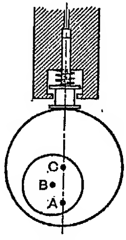

Fig. 111.

A, B, and C are the respective centres of the shaft which operates the pump, a disc fixed to the shaft, and an eccentric sheave mounted on the disc. The sheave can be turned round the disc by the governor mechanism and in this way the throw of the cam can be changed. The stroke of the pump is small compared with the total throw of the cam, and the return stroke, which is effected by means of a spring, is limited by the stops shown in the figure. AB and BC are each equal to 1 inch, and the stroke of the pump is $\frac{1}{4}$ inch when the angle ABC is 120°.

Through what angle must the sheave be turned in order to reduce the stroke to $\frac{1}{8}$ inch? By what fraction of a revolution of the shaft will the period of injection be retarded? [B.E.]

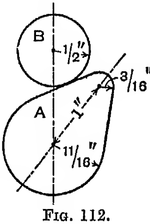

Fig. 112.

7. A petrol engine, working on the usual Otto cycle, has a straight-sided valve cam A of the form shown in Fig. 112: the roller B is constrained to move in a vertical path. The mass of the roller together with all the parts which move with it is 8 oz. Show that when the engine is running at 2400 revolutions per minute there is a sudden

change of acceleration equal to about 8000 feet per second per second, at the instant when the roller passes from the straight flank on to the curve of smaller radius, and that a valve spring giving a force of over 61 lb. weight is necessary if the roller is to remain in contact with the cam throughout the lift. [C.U.]

8. A cam rotates with uniform angular velocity ω about a fixed centre A and operates a roller whose centre C moves on a straight line through A, the point of contact being P. If B is the centre of curvature of the cam at P, show that the acceleration of C is given by ω^2AZ where Z is obtained as follows: CB meets the line through A perpendicular to CA in N. K is a point on CB such that $BN^2 = BC.BK$ and KZ is dr[illegible] CB to meet CA in Z. Show also that in the [illegible] the cam surface at the point of contact is flat, KN = NC. [C.U.]

9. A cam rotating uniformly about A (Fig. 113), with an angular velocity ω, gives a reciprocating motion in a vertical line through A to a rod PF, the rod being kept in contact with the cam by a spring. The part PD of the profile of the cam is an involute of the circle with radius AC, the part EP is straight, and the angles EPC and PCA are right angles. Prove that in the position shown the change of acceleration of P due to the change of curvature of the cam is given by $\omega^2 AP \sec^4 \theta$. [C.U.]

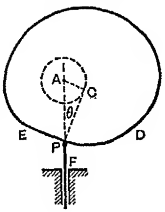

FIG. 113.

10. The fuel pump of a two-stroke oil engine is operated by a cam on the engine crankshaft. The tappet clearance is such that the cam makes contact during 60° of the revolution of the engine crank. The motion of the pump plunger during this time is simple harmonic. The plunger and its attached parts weigh $\frac{1}{2}$ lb., the stroke is $\frac{1}{2}$ inch, and the oil pressure produces a load of 80 lb. on the plunger at the end of its stroke. What is the greatest force that must be exerted by the spring in order to maintain contact between cam and roller when the engine runs at 900 r.p.m.? [U.L.]

11. As explained in Art. 72, the curves 1 and 2 in Figs. 106 to 108 concern a concave cam with roller follower, and the dimensions of the mechanism, in millimetres, are as follows: $r_1 = 14{\cdot}75$, $r_2 = 7{\cdot}5$, $r_3 = 35{\cdot}0$, $r_4 = 6{\cdot}0$, and $r_5 = 24{\cdot}5$. Show that $\theta = 22° \ 6'$ at the points B on the curves, and calculate the corresponding lift of the tappet.

12. Referring to the preceding exercise and Fig. 108, calculate the acceleration at the point B_2, taking the crankshaft speed to be 1400 r.p.m. and $\theta_2 = 51° \ 53'$; then assuming that the weight of the valve and tappet, etc., is $\frac{3}{4}$ lb., find the least spring force which is required at this point to prevent the tappet leaving the cam.

CHAPTER IX

MOTION OF RIGID BODIES IN TWO DIMENSIONS

74. Definitions—d'Alembert's Principle.—If a force P is applied to a particle of mass m and gives it an acceleration f, the direction of the force and the acceleration being the same, then $P = mf$ (Fig. 114). The force P is called the *applied force* and the product mf is called the *effective force*. If the effective force were reversed, then it would be in equilibrium with the applied force.

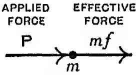

Fig. 114.

A *rigid body* may be defined as one in which the distances between the particles of which it is composed remain unchanged by the action of applied forces. Actually there is no such thing as a perfectly rigid body, but in many practical problems it is sufficiently accurate to regard a body as being rigid.

When applied forces accelerate a rigid body there are also internal forces and effective forces acting on every particle of the body. By what is known as *d'Alembert's principle* the internal forces are in equilibrium amongst themselves, and the applied forces are in equilibrium with the *reversed* effective forces. Alternatively, it may be said that the effective forces form a system which is equivalent to the system of applied forces.

In the figures in this chapter the effective forces are not shown reversed, but there will be no difficulty in distinguishing between an applied force and an effective force because the latter is the product of a mass and an acceleration.

To save space the fluxional notation is used for denoting velocity and acceleration (see Art. 6, p. 6).

75. Equations of Motion—Considering Applied and Effective Forces.—Suppose a body BC (Fig. 115), of mass M, moves due to the action of applied forces P_1, P_2, P_3, etc., then it will be shown that—

(1) The centre of gravity G moves as though the whole mass M were collected at G and all the applied forces were transferred to G with their lines of action parallel to their given lines.

(2) The body turns about its centre of gravity G, under the action of the given forces, as though G were fixed.

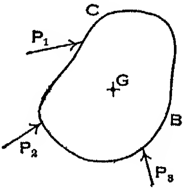

FIG. 115.

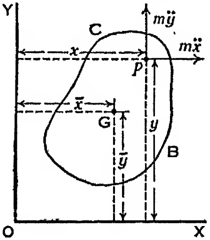

FIG. 116.

Let the co-ordinates of the centre of gravity G of the body BC (Fig. 116) be $\bar{x}$ and $\bar{y}$, at time t, with respect to fixed axes OX and OY, and let the co-ordinates of a particle p of mass m be x and y with respect to the same axes.

The components of the effective forces acting on the particle p, parallel to the axes OX and OY, are respectively

$$m\ddot{x} \quad \text{and} \quad m\ddot{y}.$$

Therefore, if the sums of the components of all the applied forces are P and Q, acting in directions parallel to OX and OY respectively,

$$P = \Sigma m\ddot{x} \quad \text{and} \quad Q = \Sigma m\ddot{y},$$

where $\Sigma m\ddot{x}$ denotes the sum of all the effective forces acting parallel to OX, and $\Sigma m\ddot{y}$ denotes the sum of all the effective forces acting parallel to OY.

Since G is the centre of gravity of the body,

$$\Sigma mx = M\bar{x} \quad \text{and} \quad \Sigma my = M\bar{y}.$$

Differentiating twice with respect to time,

$$\Sigma m\ddot{x} = M\ddot{\bar{x}} \quad \text{and} \quad \Sigma m\ddot{y} = M\ddot{\bar{y}}.$$

Therefore $$P = M\ddot{\bar{x}} \quad \text{and} \quad Q = M\ddot{\bar{y}} \quad . \quad . \quad (1),$$

and the centre of gravity G moves as though the whole mass were collected at G and the components P and Q of the applied forces were transferred to G.

Now let axes GX′ and GY′ be drawn parallel to OX and OY, respectively (Fig. 117), and let GX′ and GY′ move with the body but always remain parallel to the fixed axes.

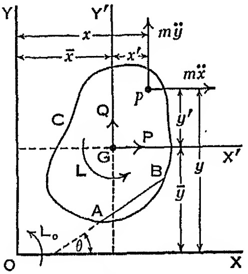

Fig. 117.

Let the co-ordinates of the particle p be x' and y' with respect to the axes GX′ and GY′, then

$$x = \bar{x} + x' \quad \text{and} \quad y = \bar{y} + y',$$

and differentiating twice with respect to time,

$$\ddot{x} = \ddot{\bar{x}} + \ddot{x}' \quad \text{and} \quad \ddot{y} = \ddot{\bar{y}} + \ddot{y}'.$$

Let the sum of the moments about G of the applied forces be L, then these applied forces may be replaced by the forces P and Q acting at G together with a couple L; also let the sum of the moments about O of the applied forces be L_O, then, taking moments about O,

$$L_O = L + Q\bar{x} - P\bar{y} \quad . \quad . \quad . \quad (2),$$

assuming the directions of the couples L and L_O to be anticlockwise.

Equating the moments about O of the applied and effective forces,

$$\begin{aligned} L_O &= \Sigma m\ddot{y}x - \Sigma m\ddot{x}y \\ &= \Sigma m\{(\ddot{\bar{y}} + \ddot{y}')(\bar{x} + x') - (\ddot{\bar{x}} + \ddot{x}')(\bar{y} + y')\} \\ &= \Sigma m\{(\ddot{\bar{y}}\bar{x} + \ddot{y}'\bar{x} + \ddot{\bar{y}}x' + \ddot{y}'x') - (\ddot{\bar{x}}\bar{y} + \ddot{x}'\bar{y} + \ddot{\bar{x}}y' + \ddot{x}'y')\} \quad (3). \end{aligned}$$

Now $\Sigma m = \mathrm{M}$ and since G is the centre of gravity, $\Sigma mx' = 0$ and $\Sigma my' = 0$; also, differentiating twice, $\Sigma m\ddot{x}' = 0$ and $\Sigma m\ddot{y}' = 0$. Therefore equation (3) becomes

$$\mathrm{L_O} = \mathrm{M}(\ddot{\bar{y}}\bar{x} - \ddot{\bar{x}}\bar{y}) + \Sigma m(\ddot{y}'x' - \ddot{x}'y') \quad . \quad . \quad (4).$$

But from (2) and (1)

$$\mathrm{L_O} = \mathrm{L} + \mathrm{Q}\bar{x} - \mathrm{P}\bar{y} = \mathrm{L} + \mathrm{M}(\ddot{\bar{y}}\bar{x} - \ddot{\bar{x}}\bar{y}) \quad . \quad . \quad (5).$$

Therefore from (4) and (5), eliminating $\mathrm{L_O}$,

$$\mathrm{L} = \Sigma m(\ddot{y}'x' - \ddot{x}'y') \quad . \quad . \quad . \quad (6).$$

Therefore the sum of the moments about G of the applied forces is equal to the sum of the moments about G of the effective forces, and the body turns as though G were a fixed point, for the variables in (6) are measured with respect to the moving axes GX′ and GY′.

Equation (6) may now be put into a more convenient form by employing polar co-ordinates.

In rectangular co-ordinates the accelerations of the particle p, relative to G, are $\ddot{x}'$ and $\ddot{y}'$ (Fig. 118), and in polar co-ordinates the accelerations are $r\ddot{\phi}$ perpendicular to Gp and $r\dot{\phi}^2$ along pG, where $r = \mathrm{G}p$ and ϕ is the angle pGX′.

Therefore, taking moments about G of the effective forces,

$$m(\ddot{y}'x' - \ddot{x}'y') = mr^2\ddot{\phi} \quad . \quad (7).$$

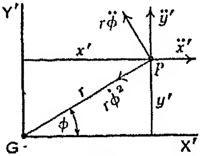

FIG. 118.

This relation could also be obtained by writing $x' = r \cos \phi$ and $y' = r \sin \phi$, differentiating twice with respect to time, and substituting in the left-hand side of (7).

At any instant the angle ϕ will not be the same for all particles in the plane X′GY′, but they will all have the same angular acceleration $\ddot{\phi}$, for if

$$\phi = \theta + \text{a constant},$$

where θ is the angle which any line AB (Fig. 117), fixed on the body, makes with the fixed axis OX at time t,

then differentiating twice with respect to t,

$$\dot{\phi}=\ddot{\theta} \quad . \quad . \quad . \quad . \quad (8),$$

and this is the angular acceleration of the body.

Therefore, from (6), (7), and (8),

$$L=\Sigma mr^2\ddot{\theta},$$

but $\Sigma mr^2=Mk^2=I$, where k is the radius of gyration and I is the moment of inertia of the body about an axis through G perpendicular to the plane X'GY', therefore

$$L=Mk^2\ddot{\theta}=I\ddot{\theta} \quad . \quad (9).$$

The results are summarized in Fig. 119, where P and Q are respectively the sums of the x and y components of the applied forces, L is the sum of the moments about G of the applied forces, $M\ddot{x}$ and $M\ddot{y}$ are the effective forces, and $Mk^2\ddot{\theta}$ is the effective couple.

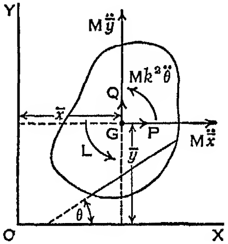

FIG. 119.

The equations are

$$P=M\ddot{x}, \quad Q=M\ddot{y}, \quad \text{and} \quad L=Mk^2\ddot{\theta} \quad . \quad (10).$$

If it is required to take moments about any point other than G, say O, then if L_O is the resultant anticlockwise moment about O of the applied forces,

$$L_O=M\ddot{y}\bar{x}-M\ddot{x}\bar{y}+Mk^2\ddot{\theta} \quad . \quad . \quad (11).$$

In the Figure the arrow-heads on the effective forces, and on the effective couple, point in the directions in which the variables increase.

Since one of the two fixed axes OX and OY may be drawn in any direction, it follows that the applied forces may be resolved parallel to any direction and the sum of their components may be equated to the sum of the components of the effective forces in the same direction. The examples which follow will make the method clear.

Example 1.—A lorry weighing 2·5 tons when loaded has its centre of gravity 36 inches in front of the back axle

centre line, 78 inches behind the front axle centre line, and 40 inches above the road. Given that its speed is 25 miles per hour on a horizontal road and that the coefficient of adhesion is 0·5, it is required to find the minimum distance in which the lorry can be stopped: (*a*) using front brakes; (*b*) using rear brakes; (*c*) using front and rear brakes, assuming that both sets of brakes are applied simultaneously.

(*a*) *Front Brakes.*—Denoting the retardation by f, the effective force is $Mf = \frac{W}{g}f$ acting through G, the centre of gravity, and parallel to the road. The applied forces are the weight W, the reactions R_1 and R_2 at the wheels (Fig. 120), and the horizontal force $0{\cdot}5R_1$ on the front wheels at the road level. (Of course R_1, R_2, and $0{\cdot}5R_1$ are each shared by two wheels.) The wheelbase $AB = 36 + 78 = 114$ inches.

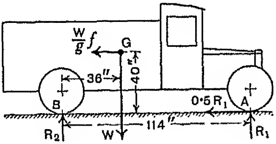

Fig. 120.

Equating applied and effective forces acting horizontally,

$$0{\cdot}5R_1 = \frac{W}{g}f \quad . \quad . \quad . \quad (1).$$

Equating the moments about B of the applied and effective forces,

$$114R_1 - 36W = 40\frac{W}{g}f \quad . \quad . \quad (2).$$

From (1) and (2), eliminating R_1,

$$114 \times \frac{2W}{g}f - 36W = 40\frac{W}{g}f,$$

then $$f = \frac{36 \times 32{\cdot}2}{188} = 6{\cdot}17 \text{ ft./sec.}^2.$$

Care should be taken with the units. In (2) the moments

are inches times force, but the result would be the same if each side of the equation were divided by 12. Since g is taken as 32·2, the units of f must be feet and seconds. Note that the value of W does not affect the value of f, but it would be required if R_1 and R_2 had to be determined.

For uniform retardation, $v^2 = 2fs$ or $s = v^2/2f$, where v is the initial velocity and s is the distance travelled.

$$\text{Now} \quad v = 25 \text{ miles per hour} = 25 \times \frac{88}{60} = \frac{110}{3} \text{ ft./sec.,}$$

$$\text{therefore} \quad s = \left(\frac{110}{3}\right)^2 \times \frac{1}{2 \times 6{\cdot}17} = 109 \text{ feet.}$$

(*b*) *Rear Brakes.*—The horizontal applied force is now $0{\cdot}5R_2$ acting at B (Fig. 121).

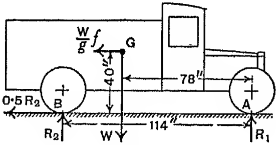

FIG. 121.

Equating applied and effective forces acting horizontally,

$$0{\cdot}5R_2 = \frac{W}{g}f \quad . \quad . \quad . \quad (1).$$

Equating moments about A of the applied and effective forces,

$$78W - 114R_2 = 40\frac{W}{g}f \quad . \quad . \quad (2).$$

From (1) and (2)

$$f = \frac{78 \times 32{\cdot}2}{268} = 9{\cdot}37 \text{ ft./sec.}^2.$$

$$\text{Then} \quad s = \frac{v^2}{2f} = \left(\frac{110}{3}\right)^2 \times \frac{1}{2 \times 9{\cdot}37} = 71{\cdot}7 \text{ feet.}$$

(c) *Front and Rear Brakes.*—The forces are as shown in Fig. 122.

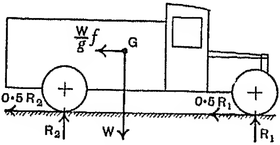

FIG. 122.

Equating applied and effective forces acting horizontally,

$$0{\cdot}5R_1 + 0{\cdot}5R_2 = \frac{W}{g}f \quad . \quad . \quad . \quad (1).$$

Equating applied and effective forces acting vertically, the latter forces being zero,

$$R_1 + R_2 - W = 0 \quad . \quad . \quad . \quad (2).$$

From (1) and (2) $\qquad 0{\cdot}5W = \dfrac{W}{g}f,$

therefore $\qquad f = 0{\cdot}5 \times 32{\cdot}2 = 16{\cdot}1$ ft./sec.^{2}.

Then $\qquad s = \dfrac{v^2}{2f} = \left(\dfrac{110}{3}\right)^2 \times \dfrac{1}{2 \times 16{\cdot}1} = 41{\cdot}8$ feet.

Example 2.—A pair of wheels and an axle, weighing 1500 lb. and having a radius of gyration $k = 1{\cdot}3$ feet, start from rest and travel on greasy rails down a uniform incline which makes an angle $\beta = 10°$ with the horizontal (Fig. 123). Each wheel has a radius $r = 1{\cdot}5$ feet, and the coefficient of friction between the wheels and the rails is $\mu = 0{\cdot}07$.

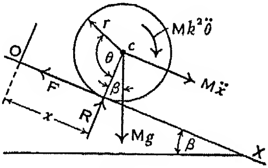

FIG. 123.

It is required to find if slip will occur and to calculate the time taken to travel 300 feet down the incline.

Let the contact be at O when the motion begins, and assume that at time t the centre c has travelled a distance x and the wheels have turned through an angle θ.

Let M be the total mass and let R and F be the normal reaction and the frictional force, respectively, between the wheels and the rails. (Actually $\frac{1}{2}$R and $\frac{1}{2}$F will act on each wheel.) At time t the effective force is $M\ddot{x}$, the effective couple is $Mk^2\ddot{\theta}$, and the applied forces are R, F, and Mg, all acting as shown. Taking $M = W/g$, the value of Mg is $W = 1500$ lb.

If there is no slip,

$$x = r\theta \quad \text{and therefore} \quad \ddot{x} = r\ddot{\theta} \qquad . \qquad (1).$$

If slip occurs, or is about to occur,

$$F = \mu R \quad . \quad . \quad . \quad . \qquad (2).$$

Equating applied and effective forces by resolving perpendicular to the track,

$$R - Mg\cos\beta = 0 \quad . \quad . \quad . \qquad (3),$$

since there is no effective force in this direction.

Equating applied and effective forces by resolving parallel to the track,

$$Mg\sin\beta - F = M\ddot{x} \quad . \quad . \quad . \qquad (4).$$

Equating moments about c of applied and effective forces,

$$Fr = Mk^2\ddot{\theta} \quad . \quad . \quad . \qquad (5).$$

It should be noted that equations (3) to (5) are true whether there is slip or pure rolling.

From (4) and (5),

$$\ddot{x} = g\sin\beta - \frac{k^2}{r}\ddot{\theta} \quad . \quad . \quad . \qquad (6).$$

Assuming no slip.

Since $\ddot{x} = r\ddot{\theta}$, substituting in (6) and solving for $\ddot{\theta}$,

$$\ddot{\theta} = \frac{gr\sin\beta}{k^2 + r^2}.$$

Then from (5), $F=\frac{Mk^2}{r}\ddot{\theta}=\frac{Mgk^2 \sin\beta}{k^2+r^2}$.

Now $Mg=1500$ lb., $k=1\cdot3$ ft., $\sin\beta=\sin 10^\circ=0\cdot1736$, and $r=1\cdot5$ ft., therefore

$$F=\frac{1500\times1\cdot3^2\times0\cdot1736}{1\cdot3^2+1\cdot5^2}=112 \text{ lb.}$$

Assuming slip.

From (2) and (3),

$$F=\mu R=\mu Mg\cos\beta \quad . \quad . \quad . \quad (7).$$

Since $\mu=0\cdot07$ and $\cos\beta=\cos 10^\circ=0\cdot9848$, therefore

$$F=0\cdot07\times1500\times0\cdot9848=103 \text{ lb.}$$

This value is less than 112 lb., therefore slip occurs. Owing to the low coefficient of friction, the force F does not reach the value which is large enough to cause pure rolling.

From (4) $\qquad \ddot{x}=g\sin\beta-\frac{F}{M}$,

and from (7) $\qquad \frac{F}{M}=\mu g\cos\beta$,

therefore $\qquad \ddot{x}=g(\sin\beta-\mu\cos\beta)$.

Substituting numerical values,

$$\ddot{x}=32\cdot2(0\cdot1736-0\cdot07\times0\cdot9848)=3\cdot37 \text{ ft./sec.}^2.$$

If s is the distance travelled in time t with constant acceleration $\ddot{x}$, then $s=\frac{1}{2}\ddot{x}t^2$, or $t=\sqrt{(2s/\ddot{x})}$.

Putting $s=300$ feet, $t=\sqrt{\frac{2\times300}{3\cdot37}}=13\cdot3$ sec.

76. Compound Pendulum.—A body of mass M swings about an axis O which is horizontal and perpendicular to the plane of the paper (Fig. 124). The centre of gravity is at G, the length $OG=a$, and the radius of gyration about G is k. Assuming the swings to be small, it is required to find the periodic time and the length of the equivalent simple pendulum.

Let OG make an angle θ with the vertical OY at time t, then if G has travelled a distance s from its mean position, $s=a\theta$ and, differentiating twice with respect to time, the acceleration $\ddot{s}=a\ddot{\theta}$.

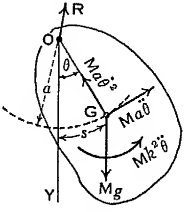

FIG. 124.

The accelerations of G are $a\dot{\theta}^2$ towards O and $a\ddot{\theta}$ perpendicular to OG, and the corresponding effective forces are $Ma\dot{\theta}^2$ and $Ma\ddot{\theta}$. The effective couple is $Mk^2\ddot{\theta}$.

The applied forces are Mg acting vertically downwards at G and the reaction R at O.

Equating the moments about O of the effective and applied forces and so avoiding the unknown reaction at O,

$$Mk^2\ddot{\theta}+Ma^2\ddot{\theta}=-Mga\sin\theta,$$

therefore $$\ddot{\theta}(k^2+a^2)+ga\sin\theta=0,$$

or $$\ddot{\theta}+\frac{ga}{k^2+a^2}\sin\theta=0 \quad . \quad . \quad . \quad (1).$$

For small values of θ, $\sin\theta=\theta$ approximately, then

$$\ddot{\theta}+\frac{ga}{k^2+a^2}\theta=0 \quad . \quad . \quad . \quad (2).$$

This equation represents simple harmonic motion (Art. 60, p. 104) and may be written as

$$\ddot{\theta}+\omega^2\theta=0, \quad \text{where} \quad \omega=\sqrt{\frac{ga}{k^2+a^2}}.$$

The periodic time is

$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{k^2+a^2}{ga}} \quad . \quad . \quad (3).$$

Now the periodic time of a simple pendulum (Art. 61) is $2\pi\sqrt{(l/g)}$ where l is its length. Therefore a simple pendulum of length $l=(k^2+a^2)/a$ will have the same periodic time as the compound pendulum and it is called the *equivalent simple pendulum.*

A closer approximation to the value of the periodic time may be obtained in the manner shown in Art. 63, p. 112. For the compound pendulum

$$T = 2\pi\sqrt{\frac{k^2 + a^2}{ga}} \times F,$$

where F is the amplitude factor.

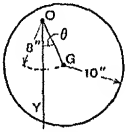

FIG. 125.

Example.—The body is a disc (Fig. 125) of radius $r = 10$ inches and $OG = a = 8$ inches. The horizontal axis through O is perpendicular to the face of the disc and the swings are small. To find l and T.

$$k^2 = \frac{r^2}{2} = \frac{10^2}{2} = 50 \text{ in.}^2, \qquad l = \frac{k^2 + a^2}{a} = \frac{50 + 64}{8} = 14{\cdot}25 \text{ in.}$$

$$T = 2\pi\sqrt{\frac{k^2 + a^2}{ga}} = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{14{\cdot}25}{32{\cdot}2 \times 12}} = 1{\cdot}21 \text{ sec.}$$

77. Centre of Oscillation.—As shown in the preceding Art., the periodic time of a compound pendulum is $T = 2\pi\sqrt{\frac{k^2 + a^2}{ga}}$ and the length of the equivalent simple pendulum is $l = (k^2 + a^2)/a$.

Referring to Fig. 126, O is the centre of suspension and G is the centre of gravity. Join OG and produce to O_1, making $OO_1 = l$, then the point O_1 is called the *centre of oscillation.*

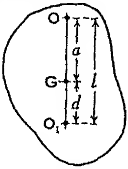

FIG. 126.

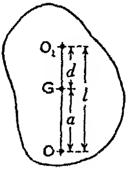

FIG. 127.

Let $O_1G = d$, then $OG + O_1G = a + d = l = (k^2 + a^2)/a$, therefore

$$a(a + d) = k^2 + a^2, \quad \text{or} \quad ad = k^2, \quad \text{or} \quad d = k^2/a.$$

It will now be shown that if the pendulum is suspended at O_1 (Fig. 127), the point O becomes the centre of oscillation.

When O_1 is the centre of suspension, $T=2\pi\sqrt{\frac{k^2+d^2}{gd}}$,

but $$\frac{k^2+d^2}{d}=\frac{k^2+(k^2/a)^2}{k^2/a}=\frac{a^2+k^2}{a}=l=OO_1,$$

therefore the point O is the centre of oscillation and the periodic time has the same value as before.

Kater's Pendulum.—Kater used a pendulum with two adjustable knife-edges and an adjustable mass for determining the value of g. The positions of the mass and the knife-edges are adjusted until the time of oscillation is the same about each knife-edge, then if l is the distance between the knife-edges, $T=2\pi\sqrt{(l/g)}$ and the value of g can be calculated.

78. Compound Pendulum Reactions.—Given that the compound pendulum (Fig. 128) has a maximum angular displacement β from its mean position, it is required to find the components R_1 and R_2 of the reaction at the support O when the displacement is θ. The directions of the components R_1 and R_2 are respectively perpendicular to and along GO.

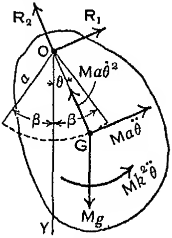

FIG. 128.

The angular acceleration and angular velocity of the pendulum will be wanted in terms of the angular displacement θ.

From Art. 76, equation (1),

$$\ddot{\theta}=-\frac{ga}{k^2+a^2}\sin\theta \quad . \quad . \quad . \quad (1).$$

Multiplying each side by $2\dot{\theta}$,

$$2\dot{\theta}\ddot{\theta}=-\frac{2ga}{k^2+a^2}\sin\theta\,\dot{\theta}.$$

Integrating, $$\dot{\theta}^2=\frac{2ga}{k^2+a^2}\cos\theta+C,$$

where C is a constant of integration.

When $\theta=\beta$, $\dot{\theta}=0$, therefore the constant

$$C=-\frac{2ga}{k^2+a^2}\cos\beta,$$

and
$$\dot{\theta}^2=\frac{2ga}{k^2+a^2}(\cos\theta-\cos\beta) \quad . \quad . \quad (2).$$

To find R_1, equate the moments about G of the applied and effective forces, then

$$R_1 a=-Mk^2\ddot{\theta},$$

and substituting the value of $\ddot{\theta}$ from (1) and simplifying,

$$R_1=\frac{Mgk^2}{k^2+a^2}\sin\theta \quad . \quad . \quad . \quad (3).$$

To find R_2, equate the applied and effective forces by resolving along GO, then

$$R_2-Mg\cos\theta=Ma\dot{\theta}^2,$$

and substituting the value of $\dot{\theta}^2$ from (2) and simplifying,

$$R_2=Mg\{\cos\theta+\frac{2a^2}{k^2+a^2}(\cos\theta-\cos\beta)\} \quad . \quad (4).$$

When the swings are small, $\sin\theta=\theta$ approximately and $\cos\theta=\cos\beta=1$ approximately, then

$$R_1=\frac{Mgk^2}{k^2+a^2}\theta \quad \text{and} \quad R_2=Mg$$

approximately.

It should be noted that for small swings the maximum value of the angular velocity $\dot{\theta}$ is small. This is evident from equation (2), for $\dot{\theta}^2$ is a maximum when $\theta=0$, and when β is small, $\cos 0°-\cos\beta$ is small and therefore $\dot{\theta}^2$ and $\dot{\theta}$ are small.

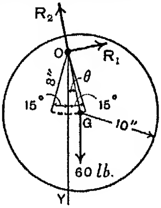

FIG. 129.

Example.—The body is a circular disc (Fig. 129) weighing 60 lb., of radius $r=10$ inches, and $OG=a=8$ inches. The horizontal axis through O is perpendicular to the face of the disc. The maximum angular displacement from the mean

position is $\beta=15°$. It is required to find the components R_1 and R_2 of the reaction at O when θ has each of the values 0°, 5°, 10°, and 15°.

Now $$R_1=\frac{Mgk^2}{k^2+a^2}\sin\theta,$$

and $$R_2=Mg\{\cos\theta+\frac{2a^2}{k^2+a^2}(\cos\theta-\cos\beta)\}.$$

Mass $M=W/g$, therefore $Mg=W=60$ lb.

$$\cos\beta=\cos 15°=0{\cdot}9659.$$

$$k^2=\tfrac{1}{2}r^2=\tfrac{1}{2}\times 10^2=50 \text{ in.}^2. \qquad a^2=8^2=64 \text{ in.}^2.$$

$$\frac{Mgk^2}{k^2+a^2}=\frac{60\times 50}{50+64}=\frac{3000}{114}=\frac{1000}{38}\text{ lb.}$$

$$\frac{2a^2}{k^2+a^2}=\frac{2\times 64}{50+64}=\frac{64}{57}.$$

Therefore $$R_1=\frac{1000}{38}\sin\theta \text{ lb.}$$

and $$R_2=60\left\{\cos\theta+\frac{64}{57}(\cos\theta-0{\cdot}9659)\right\}\text{ lb.}$$

Substituting the given values of θ, then it is found that R_1 and R_2 have the values shown in the table.

θ	0°	5°	10°	15°
R_1 lb.	0	2·29	4·57	6·81
R_2 lb.	62·3	61·8	60·4	58·0

79. **Work and Energy.**—From Art. 75, equations (10),

$$P=M\ddot{x},\quad Q=M\ddot{y},\quad\text{and}\quad L=Mk^2\ddot{\theta}.$$

Denoting the velocities $\dot{x}$, $\dot{y}$, and $\dot{\theta}$ by u, v, and ω respectively, then

$$P=M\dot{u},\quad Q=M\dot{v},\quad\text{and}\quad L=Mk^2\dot{\omega}\qquad . \quad (1),$$

as indicated in Fig. 130.

Suppose that in time δt the centre of gravity of the body has small displacements $\delta\bar{x}$ and $\delta\bar{y}$, and the body has a small angular displacement $\delta\theta$, then

Work done by applied forces = Work done by effective forces.

$$P\delta\bar{x} + Q\delta\bar{y} + L\delta\theta = M\dot{u}\delta\bar{x} + M\dot{v}\delta\bar{y} + Mk^2\dot{\omega}\delta\theta.$$

Now $\qquad \delta\bar{x} = u\delta t \quad$ and $\quad \dot{u}\delta t = \delta u,$

therefore $\qquad M\dot{u}\delta\bar{x} = M\dot{u}u\delta t = Mu\delta u.$

Similarly,

$$M\dot{v}\delta\bar{y} = Mv\delta v \quad \text{and} \quad Mk^2\dot{\omega}\delta\theta = Mk^2\omega\delta\omega.$$

Therefore

$$P\delta\bar{x} + Q\delta\bar{y} + L\delta\theta = Mu\delta u + Mv\delta v + Mk^2\omega\delta\omega.$$

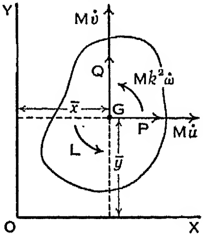

Fig. 130.

If during time t the displacements are $\bar{x}_1 - \bar{x}_0$, $\bar{y}_1 - \bar{y}_0$, and $\theta_1 - \theta_0$, and the corresponding velocity changes are $u_1 - u_0$, $v_1 - v_0$, and $\omega_1 - \omega_0$, then integrating between the appropriate limits,

$$\int_{\bar{x}_0}^{\bar{x}_1} Pd\bar{x} + \int_{\bar{y}_0}^{\bar{y}_1} Qd\bar{y} + \int_{\theta_0}^{\theta_1} Ld\theta = \int_{u_0}^{u_1} Mudu + \int_{v_0}^{v_1} Mvdv + \int_{\omega_0}^{\omega_1} Mk^2\omega d\omega$$

$$= \tfrac{1}{2}M(u_1^2 - u_0^2) + \tfrac{1}{2}M(v_1^2 - v_0^2) + \tfrac{1}{2}Mk^2(\omega_1^2 - \omega_0^2)$$

$$= \text{change in kinetic energy} \quad . \quad . \quad . \quad (2).$$

Therefore during a given time the work done by the applied forces is equal to the change in the kinetic energy.

It must be understood here that the work done by the

applied forces is to be interpreted as the work done by the forces which produce acceleration or retardation. For example, if a force P pushes a mass along a straight path against a frictional resistance F, then it is the work done by the accelerating force P − F which is equated to the change of kinetic energy of the mass.

Example 1.—A solid circular cylinder begins to roll up a slope of 1 in 5 (measured as a sine) with a linear velocity of 6 feet per second. It is required to find the distance travelled by the cylinder before coming to rest.

Let W be the weight and r the radius of the cylinder. Let s be the distance travelled and h the corresponding rise (Fig. 131). If k is the radius of gyration about the axis of the cylinder, then $k^2 = \frac{1}{2}r^2$.

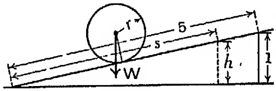

Fig. 131.

Denoting the initial linear velocity by v and the initial angular velocity by ω, then $\omega = v/r$.

The work done in lifting the cylinder a distance h is Wh.

The initial kinetic energy is made up of two parts—

(1) Kinetic Energy of Translation $= \dfrac{W}{2g}v^2$.

(2) Kinetic Energy of Rotation $= \dfrac{W}{2g}k^2\omega^2$.

Therefore the total Kinetic Energy $= \dfrac{W}{2g}(v^2 + k^2\omega^2)$.

Work done = Loss of Kinetic Energy,

therefore $$Wh = \frac{W}{2g}(v^2 + k^2\omega^2)$$

or $$h = \frac{1}{2g}\left(v^2 + \frac{r^2}{2}\frac{v^2}{r^2}\right) = \frac{3v^2}{4g}.$$

Substituting numerical values,

$$h = \frac{3 \times 6^2}{4 \times 32 \cdot 2} = 0 \cdot 839 \text{ foot}$$

and $$s = 5h = 5 \times 0 \cdot 839 = 4 \cdot 20 \text{ feet}.$$

Example 2.—A railway truck is being lowered down a gradient by means of a wire rope coiled on a winding drum at the top of the incline, the rope being supported parallel to the incline on frictionless rollers.

The total mass of the truck is M, the moment of inertia of each of its two pairs of wheels and axles is mk^2, and the radius of the wheels is r. The moment of inertia of the drum excluding the wire rope is I, and its radius is R. The total length of the wire is l, and its mass is ρ per unit length. The angle of the incline is α.

Consider the case when the truck is descending the incline, and winding wire off the drum which is perfectly free to rotate. Show that, when the length of wire paid off from the drum is s, the acceleration of the truck is

$$\frac{(\mathrm{M}+\rho s)g\sin\alpha}{\mathrm{M}+\rho l+\dfrac{2mk^2}{r^2}+\dfrac{\mathrm{I}}{\mathrm{R}^2}}. \qquad \text{[C.U.]}$$

Let v be the velocity of the truck when the length of wire paid off is s (Fig. 132). Then the angular velocity of the wheels is v/r and the angular velocity of the drum is v/R. The mass of rope paid off is ρs and the mass of the whole rope is ρl. Also the magnitude of the linear velocity of the whole rope is v, the same as that of the truck.

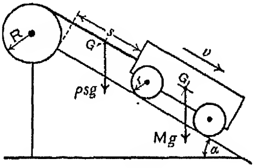

FIG. 132.

Kinetic Energy of Translation (including rope) $=\frac{1}{2}(\mathrm{M}+\rho l)v^2$.

Kinetic Energy of Rotation $=\frac{1}{2}\left(2mk^2\frac{v^2}{r^2}\right)+\frac{1}{2}\mathrm{I}\frac{v^2}{\mathrm{R}^2}$.

Total Kinetic Energy $\qquad =\frac{v^2}{2}\left\{\mathrm{M}+\rho l+\frac{2mk^2}{r^2}+\frac{\mathrm{I}}{\mathrm{R}^2}\right\} \qquad (1).$

When the truck has travelled a distance s, its centre of gravity G has been lowered a height $s\sin\alpha$, and the centre

of gravity G′ of the length s of the rope has been lowered a height $\frac{1}{2}s \sin \alpha$, therefore

$$\text{Work done by gravity} = \text{M}gs \sin \alpha + \rho sg(\tfrac{1}{2}s \sin \alpha) \qquad (2).$$

Equating (1) and (2) and solving for v^2,

$$v^2 = \frac{(2\text{M}s + \rho s^2)g \sin \alpha}{\text{M} + \rho l + \dfrac{2mk^2}{r^2} + \dfrac{\text{I}}{\text{R}^2}} \qquad (3).$$

Since v^2 depends on s^2 as well as on s, it is evident that the formula $v^2 = 2fs$ cannot be used to obtain the acceleration f. The acceleration is obtained by differentiating, with respect to t, each side of (3), then

$$2v\frac{dv}{dt} = \frac{(2\text{M} + 2\rho s)\dfrac{ds}{dt}g \sin \alpha}{\text{M} + \rho l + \dfrac{2mk^2}{r^2} + \dfrac{\text{I}}{\text{R}^2}} \qquad (4).$$

Now $\frac{ds}{dt} = v$ and dividing (4) by $2v$ gives

$$\text{Acceleration } \frac{dv}{dt} = \frac{(\text{M} + \rho s)g \sin \alpha}{\text{M} + \rho l + \dfrac{2mk^2}{r^2} + \dfrac{\text{I}}{\text{R}^2}} \qquad (5).$$

80. Impulse, Momentum, and Moment of Momentum.—From Art. 75, equations (10),

$$\text{P} = \text{M}\ddot{x}, \qquad \text{Q} = \text{M}\ddot{y}, \quad \text{and} \quad \text{L} = \text{M}k^2\ddot{\theta},$$

or using the notation of Art. 79, equations (1), and referring to Fig. 133,

$$\text{P} = \text{M}\dot{u}, \quad \text{Q} = \text{M}\dot{v}, \quad \text{and} \quad \text{L} = \text{M}k^2\dot{\omega}$$

or
$$\text{P} = \text{M}\frac{du}{dt}, \quad \text{Q} = \text{M}\frac{dv}{dt}, \quad \text{and} \quad \text{L} = \text{M}k^2\frac{d\omega}{dt} \qquad (1).$$

Let the velocities be u_0, v_0, and ω_0 at time t_0, and u_1, v_1, and ω_1 at time t_1, then integrating (1),

$$\int_{t_0}^{t_1} \text{P}dt = \text{M}\int_{u_0}^{u_1} du = \text{M}(u_1 - u_0), \qquad \int_{t_0}^{t_1} \text{Q}dt = \text{M}(v_1 - v_0),$$

and
$$\int_{t_0}^{t_1} \text{L}dt = \text{M}k^2(\omega_1 - \omega_0) \qquad (2).$$

$\int_{t_0}^{t_1} \mathrm{P}dt$ is the *impulse* of the force P during the time $t_1 - t_0$ and is equal to the *change of momentum* $\mathrm{M}(u_1 - u_0)$. Also $\int_{t_0}^{t_1} \mathrm{Q}dt$ is the impulse of the force Q and is equal to the change of momentum $\mathrm{M}(v_1 - v_0)$.

$\int_{t_0}^{t_1} \mathrm{L}dt$ is the impulse of the couple or torque L and is equal to the *change of angular momentum*, or *change of the moment of momentum*, $\mathrm{M}k^2(\omega_1 - \omega_0)$.

The expression $\mathrm{M}k^2(\omega_1 - \omega_0)$ is the change of the moment of momentum about the centre of gravity of the body, and in many cases the change of the moment of momentum about some other point is required, say about O (Fig. 133).

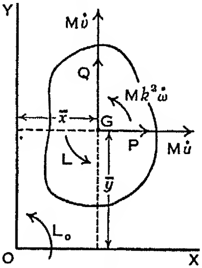

FIG. 133.

Let $\mathrm{L_O}$ be the moment about O of the applied forces, then

$$\begin{aligned}\mathrm{L_O} &= \mathrm{Q}\bar{x} - \mathrm{P}\bar{y} + \mathrm{L} \\ &= \mathrm{M}\dot{v}\bar{x} - \mathrm{M}\dot{u}\bar{y} + \mathrm{M}k^2\dot{\omega} \\ &= \frac{d}{dt}\{\mathrm{M}v\bar{x} - \mathrm{M}u\bar{y} + \mathrm{M}k^2\omega\} \qquad (3).\end{aligned}$$

(Checking by differentiation,

$$\frac{d}{dt}\{\mathrm{M}v\bar{x}\} = \mathrm{M}v\dot{\bar{x}} + \mathrm{M}\dot{v}\bar{x} = \mathrm{M}vu + \mathrm{M}\dot{v}\bar{x},$$

$$-\frac{d}{dt}\{\mathrm{M}u\bar{y}\} = -\mathrm{M}u\dot{\bar{y}} - \mathrm{M}\dot{u}\bar{y} = -\mathrm{M}uv - \mathrm{M}\dot{u}\bar{y},$$

and the terms $\mathrm{M}vu - \mathrm{M}uv$ cancel.)

Equation (3) shows that the sum of the moments of the applied forces about any fixed point is equal to the rate of change of the moment of momentum about the same point.

Also, integrating over the time interval $t_1 - t_0$,

$$\int_{t_0}^{t_1} \mathrm{L_O}dt = \text{Change in the moment of momentum during the time } t_1 - t_0 \quad . \qquad (4).$$

When two bodies collide, or when a point on a moving body becomes fixed suddenly, there is an impulsive force (see also Art. 41, p. 72) which acts for a very short time and usually its mean value is large compared with the value of any ordinary force which may be acting at the same time. In most cases the mean value of an impulsive force cannot be determined, but its impulse is equal to the change of momentum produced and this can be measured. When considering the effect of an impulsive force, any ordinary force acting at the same time may generally be disregarded.

The theorems given below follow from equations (2) and (4); they have already been given in Art. 42, p. 73, but are of sufficient importance to be restated here.

Conservation of Linear Momentum.—If, in any direction, the sum of the components of the applied forces acting on a system of bodies is zero, the total momentum of the system is constant in that direction.

Conservation of Moment of Momentum or Angular Momentum.—If, in a system of bodies, the sum of the moments of the applied forces about any fixed axis is zero, the moment of momentum of the system about that axis is constant.

Example.—In Fig. 134, AB and CB are inclined planes. A solid cylinder is placed on AB and released, its axis being at right angles to the line of maximum slope. Show that it will finally come to rest after a time $30\sqrt{\frac{h}{3g}}$ where h is the initial height of the centre above its position when the cylinder is at rest touching both planes. [C.U.]

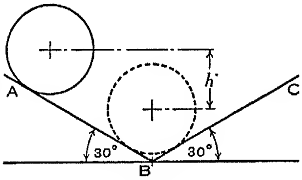

FIG. 134.

Let M be the mass of the cylinder, then Mg is its weight. Let k be the radius of gyration of the cylinder about its axis and let r be its radius, then $k^2 = \frac{1}{2}r^2$.

Descending the Left-hand Slope.—Let v be the velocity of the centre O just before impact occurs at c (Fig. 135) and let ω be the corresponding angular velocity, then $\omega = v/r$.

Work done in descending height h = Gain in kinetic energy.

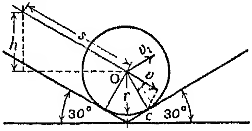

FIG. 135.

$$Mgh = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}Mk^2\omega^2$$

$$= \tfrac{1}{2}Mv^2 + \tfrac{1}{2}M\frac{r^2}{2}\frac{v^2}{r^2}.$$

Therefore $\qquad h = \dfrac{3v^2}{4g} \quad$ or $\quad v = 2\sqrt{\dfrac{gh}{3}}.$

Let s be the distance travelled by the centre O and let t be the time taken, then

$$s = \frac{h}{\sin 30^\circ} = \frac{h}{0{\cdot}5} = 2h; \quad \text{also} \quad s = \frac{v}{2}t; \quad \text{therefore} \quad \frac{vt}{2} = 2h.$$

Therefore $\qquad t = \dfrac{4h}{v} = \dfrac{4h}{2}\sqrt{\dfrac{3}{gh}} = 2\sqrt{\dfrac{3h}{g}} \quad . \quad . \quad (1),$

or $\qquad t = \dfrac{4h}{v} = \dfrac{4}{v} \times \dfrac{3v^2}{4g} = \dfrac{3v}{g} \quad . \quad . \quad . \quad (2).$

Impact.—Let v_1 and ω_1 be the initial linear velocity and angular velocity respectively up the right-hand slope, then $\omega_1 = v_1/r$.

The moment of momentum about c, the point of impact (actually a line of impact), is unchanged, because the impulsive force acting at c cannot have any moment about c; therefore

$$Mv_1r + Mk^2\omega_1 = Mvr \sin 30^\circ + Mk^2\omega,$$

$$v_1r + \frac{r^2}{2}\frac{v_1}{r} = \frac{vr}{2} + \frac{r^2v}{2r},$$

from which $\qquad v_1 = \dfrac{2}{3}v \quad . \quad . \quad . \quad . \quad (3).$

Ascending the Right-hand Slope.—Let t_1 be the time

taken to travel up the right-hand slope, then it follows from (2) that

$$t_1 = \frac{3v_1}{g}$$

or

$$\frac{t_1}{t} = \frac{v_1}{v}.$$

But from (3), $v_1 = \frac{2}{3}v$, therefore $\frac{t_1}{t} = \frac{2}{3}$ or $t_1 = \frac{2}{3}t$.

The cylinder will take an equal time t_1 to descend the right-hand slope, therefore the time for the return journey on this slope is $2t_1$.

To obtain the Total Time.—If $2t_2$, $2t_3$, $2t_4$, and so on, are the times of subsequent return journeys, then it follows that $t_2 = \frac{2}{3}t_1 = \left(\frac{2}{3}\right)^2 t$, $t_3 = \frac{2}{3}t_2 = \left(\frac{2}{3}\right)^3 t$, and so on.

If T is the total time before the cylinder comes to rest, then

$$\begin{aligned} T &= t + 2t_1 + 2t_2 + 2t_3 + \dots \\ &= (2t + 2t_1 + 2t_2 + 2t_3 + \dots) - t \\ &= 2t\left\{1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots\right\} - t. \end{aligned}$$

The series in the brackets is a geometrical progression and its sum is $\frac{1}{1-\frac{2}{3}} = 3$.

Therefore $T = 2t \times 3 - t = 5t$,

but from (1), $t = 2\sqrt{\frac{3h}{g}}$,

therefore $T = 10\sqrt{\frac{3h}{g}} = 30\sqrt{\frac{h}{3g}}$

as was to be proved.

81. Centre of Percussion.—Let G be the centre of gravity of a rigid body which is free to turn about a fixed axis

passing through O perpendicular to the plane of the paper (Fig. 136). Suppose an impulsive force P is applied to the body, its line of action being in the plane of the paper and meeting OG produced at O_1, then if there is no impulsive reaction at O, the point O_1 is called the *centre of percussion*. It should be noted, however, that the subsequent motion of the body will produce a reaction at O.

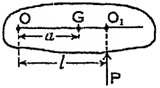

FIG. 136.

It is evident that the line of action of the impulsive force P must be perpendicular to OO_1, otherwise there would be a component parallel to OO_1 and consequently a reaction at O along OO_1.

The position of the point O_1 will now be determined. Let $OG = a$ and $OO_1 = l$; let M be the mass of the body and let k be its radius of gyration about an axis through G parallel to the axis at O. Let ω be the angular velocity of the body produced by the impulsive force P, then the velocity of G is $a\omega$ perpendicular to OG.

Suppose the force P acts for a very short time t, then equating the impulse to the change of momentum perpendicular to OO_1, remembering that there is to be no impulsive force at O,

$$Pt = Ma\omega \quad . \quad . \quad . \quad (1).$$

Taking moments about O, the impulse of the moment of the applied force is equal to the change of the moment of momentum, therefore

$$Plt = Mk^2\omega + Ma^2\omega \quad . \quad . \quad (2).$$

From (1) and (2) $\quad l = (k^2 + a^2)/a,$

and this fixes the position of O_1, the centre of percussion.

If the body were swung as a pendulum about the axis O, then the point O_1 would be the centre of oscillation, as shown in Art. 77—that is, the centre of oscillation of a pendulum is also the centre of percussion.

Exercises IX

1. A cage weighing 1·5 tons is raised by means of a rope coiled round a drum of 5 feet diameter mounted on a horizontal shaft. The drum and shaft weigh 2000 pounds and their radius of gyration is 28 inches. A motor supplies a constant torque of 9000 pound-feet to the shaft. Assuming that the rope is tight when the motor begins to revolve, find

(*a*) The acceleration of the cage;
(*b*) The time required to raise it 40 feet from rest;
(*c*) The tension of the rope.

What torque must be applied to the shaft in order that the cage may descend at a uniform speed of 2 feet per second? Neglect friction. [B.E.]

2. The loaded cage of a goods hoist is raised by a rope which passes round a drum and is connected to a balance weight at the other end. The loaded cage weighs 1·5 tons and the balance weight is 1·1 tons (Fig. 137); the drum weighs 1000 lb., its diameter is 3 feet 6 inches, and its radius of gyration is 16 inches.

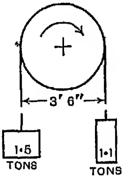

FIG. 137.

Calculate the torque which must be applied to the drum to raise the cage with an acceleration of 4 feet per second per second. What horse-power is required to give this acceleration at the instant the speed is 8 feet per second? Friction of bearings is to be neglected.

3. The centre of gravity of a motor car is at a height h above the road level, at a distance a behind the front axle and at a distance b in front of the back axle. The coefficient of friction between the tyres and road is μ. Neglecting rotational inertia, calcula[illegible]lation which can be produced by the brakes [illegible] wheels on a level road

(1) If the front wheels only are braked;
(2) If the rear wheels only are braked.

Prove that front-wheel braking will be the more efficient provided $a < b + \mu h$. If the car is descending an incline making an angle θ with the horizontal, show that this same condition still holds true. [C.U.]

4. A circular cylinder rolls without slipping down a plane inclined to the horizontal at an angle of 30°. (*a*) Find the linear acceleration of the cylinder; (*b*) What minimum coefficient of friction is necessary to prevent slipping? (*Note.*—$k^2 = \frac{1}{2}r^2$.)

5. A pair of wheels and an axle, weighing 1500 lb. and having a radius of gyration of 1·3 feet, start from rest and travel on rails down a uniform incline which makes an angle of 10° with

the horizontal. The radius of each wheel is 1·5 feet and the coefficient of friction between the wheels and the rails is 0·25. Prove that slip will not occur, and find the time taken to travel 300 feet.

6. A circular face-plate is oscillated about a horizontal axis through O (Fig. 138) perpendicular to its face, O being $8\frac{1}{4}$ inches from the centre of gravity G. The plate makes 100 complete oscillations or cycles in 117 seconds. Find the radius of gyration about an axis through G perpendicular to the face of the plate.

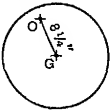

FIG. 138.

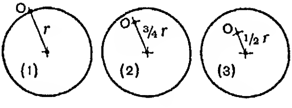

FIG. 139.

7. A flat circular plate of radius r swings in its own plane about a horizontal axis through the point O (Fig. 139). Find the length of the equivalent simple pendulum when the point O is (1) at the circumference, (2) a distance $\frac{3}{4}r$ from the centre, (3) mid-way between the centre and the circumference.

8. A flat square plate, having sides of length $2a$, swings in its own plane about a horizontal axis through a point O (Fig. 140). Find the length of the equivalent simple pendulum for each of the positions of O shown at (1), (2), and (3).

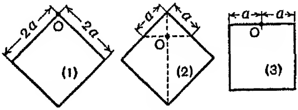

FIG. 140.

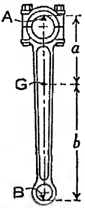

FIG. 141.

9. A connecting-rod AB (Fig. 141) was suspended on a horizontal knife-edge at A and allowed to oscillate. It was found that the rod made 60 complete oscillations or cycles per minute. Next the r ... B and made 54 cycles per minute. If the cer ... rod is a distance a from A and a distance b ... 3·75 inches, find a and b and the radius of gyration k about G in the plane of oscillation.

10. Referring to tl ... ne, if the rod makes 515 cycles in 10 min ... at A and 476 cycles in 10 minutes when ... d if $a+b=18$ inches, find a, b, and k.

11. A rectangular plate, 20 inches by 30 inches, weighing 100 lb., swings about a horizontal axis through O (Fig. 142) perpendicular to the face of the plate. If the swing is 12° on each side of the vertical OY, calculate the values of the reactions R_1 and R_2 when the angular displacement from the vertical is $\theta = 6°$. Also find the maximum value of the angular velocity.

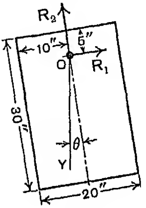

FIG. 142.

12. One end of a uniform
l is attached to the ground
other end is raised in a vertical plane until the angle made by the spar with the horizontal is θ. It is then allowed to fall under the action of
at the lower
the spar is small compared with its length, the direction of the initial reaction at the hinge makes an angle ϕ with the spar given by $\cot \phi = 4 \tan \theta$. [C.U.]

13. Two equal homogeneous solid spheres each of radius r are fixed together by a light rigid bar whose direction passes through the centres which are a distance $2l$ apart. The system is caused to oscillate as a compound pendulum about a point P in the bar distant x from the centre of the bar. Find the period of a small oscillation. Show that the period would be least if it were possible for the distance of P from the centre of the bar to be $\sqrt{\dfrac{5l^2 + 2r^2}{5}}$. Show that this point is inside one of the spheres. [U.L.]

14. A uniform disc can rotate in its own plane, which is vertical, about a smooth hinge at one end of a diameter. It is allowed to fall from the position in which this diameter is horizontal. Prove that, when the horizontal component of the reaction at the hinge is a maximum, the vertical component is $\frac{4}{3}$ of the weight of the disc. [C.U.]

15. A rigid body makes small oscillations about a horizontal axis. Prove that the periodic time is $2\pi\sqrt{k^2/hg}$, where k is the radius of gyration of the body about the axis and h the distance of the axis from the centre of gravity of the body.

A flywheel weighing 3 tons is suspended so as to oscillate about an axis perpendicular to its plane and 3 feet distant from the centre of the wheel. Find the radius of gyration of the wheel about its axis if the time of a small oscillation is 2·5 seconds, and calculate the work required when the bout its axis to increase its velocity from 50 per minute. [U.L.]

16. A solid cylinder rolls down a plane inclined at a slope of 1 vertical in 4 horizontal. Use the energy method to find the

velocity of the cylinder after it has rolled down a distance of 20 feet.

17. A four-wheeled truck is running down a slope which makes an angle β with the horizontal (Fig. 143). The total mass of the truck is M, the mass of each pair of wheels and axle is m and their radius of gyration, about their axis, is k. The radius of each wheel is r. Show that the acceleration of the truck is

$$\frac{Mg \sin \beta}{M + 2mk^2/r^2}.$$

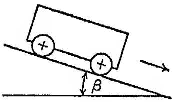

Fig. 143.

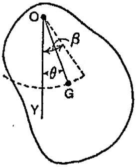

Fig. 144.

18. A compound pendulum has a maximum angular displacement β from its mean position. Show by the principle of the conservation of energy that

$$\dot{\theta}^2 = \frac{2ga}{k^2 + a^2}(\cos \theta - \cos \beta),$$

then by differentiation obtain the equation

$$\ddot{\theta} + \frac{ga}{k^2 + a^2} \sin \theta = 0,$$

where $a = OG$ (Fig. 144), the distance of the centre of gravity G from the axis of suspension O, k is the radius of gyration about an axis through G parallel to the axis of suspension, and θ is the angular displacement at time t from the vertical OY. Compare this method with that used in Art. 76, p. 141.

19. A pair of wheels and axle of mass M and moment of inertia Mk^2 stand on a horizontal surface. The radius of the wheels is a, and concentrated masses m are attached to parallel spokes at a distance b from the centre.

The wheels are held so that these spokes are turned through an angle α from their lowest position and the system is then released.

Calculate the angular velocity when these spokes make an angle θ with the vertical.

Thence prove that the period of a small oscillation is

$$2\pi \sqrt{\frac{M(a^2 + k^2) + 2m(a - b)^2}{2mgb}}. \qquad \text{[C.U.]}$$

20. Two toothed wheels (1) and (2) (Fig. 145), running freely on parallel shafts, are suddenly put into mesh. For wheel (1), radius $r_1 = 6$ inches, radius of gyration $k_1 = 4{\cdot}8$ inches, and weight $W_1 = 15$ lb. For wheel (2), $r_2 = 8$ inches, $k_2 = 6{\cdot}7$ inches, and $W_2 = 25$ lb. The speeds before meshing are $N_1 = 50$ r.p.m. and $N_2 = 30$ r.p.m. Find N_1' and N_2', the speeds after meshing, and also find the loss of energy in inch-pounds, taking N_1' and N_2' to three significant figures.

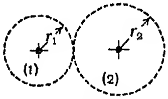

Fig. 145.

21. Using the data from the preceding exercise, but taking $N_1 = 100$ r.p.m. and $N_2 = 0$, find N_1' and N_2' and the loss of energy in inch-pounds, taking N_1' and N_2' to three significant figures.

22. Two wheels, A and B, rotate about parallel axes. Initially the wheels are not in contact; A rotates at 300 r.p.m. and B is at rest. By a suitable lever arrangement the wheel B is caused to press with its rim against that of A, the normal pressure between the wheels being 400 pounds; coefficient of friction is 0·06; the moments of inertia of A and B are 3600 lb. inch² and 2000 lb. inch² respectively; diameter of each wheel is 18 inches.

Neglecting the friction of the bearings, determine (i) the angular acceleration of B and the angular retardation of A; (ii) the common angular velocity of the wheels when slipping has ceased; and the time taken to attain this common velocity. [B.E.]

23. A flywheel (A) having a moment of inertia I_A and a radius r_A is coupled by means of a belt drive to another flywheel (B) having a moment of inertia I_B and a radius r_B.

When the belt is thrown on, B is at rest and A has an angular velocity ω.

Show that when slipping of the belt has ceased, the angular velocity acquired by B is

$$\frac{r_A r_B I_A \omega}{r_B^2 I_A + r_A^2 I_B},$$

and that the time during which slip of the belt occurs is

$$\frac{I_A I_B r_A \omega}{(T_1 - T_2)(r_B^2 I_A + r_A^2 I_B)},$$

where T_1 and T_2 are the tensions of the tight and slack sides of the belt during the period of slipping. [C.U.]

24. A sphere, of radius a, rolls, with constant angular velocity ω, on a horizontal plane, directly towards a step of height h. Supposing $a > h$, find the least value of ω that will enable the sphere to mount the step; and, supposing ω greater than this,

find the angular velocity with which it will roll on the upper plane. Assume no slip or rebound when the sphere strikes the step. [B.E.]

25. A concentrated mass M is rigidly attached by a light rod to a horizontal shaft which can rotate in frictionless bearings. The mass is hanging in the equilibrium position, when there is applied to the shaft a constant axial torque D. Show that the shaft and mass will continue to rotate always in the same direction only if D be greater than $0{\cdot}725Mgl$, where l is the distance from M to the axis of the shaft.

[The equation, $\theta \sin\theta = 1 - \cos\theta$, is satisfied by $\theta = 2{\cdot}33$ radians.] [C.U.]

26. A plumb-line consisting of a small mass suspended from a string of length l is attached to the roof of a railway carriage. With the car [illegible] a straight track at a uniform speed v, the [illegible] and stationary relative to the carriage. If the carriage suddenly enters a curve of radius r, show that the plumb-line will start to oscillate, the deflection from the vertical being given by

$$\theta = \frac{v^2}{rg}\left[1 - \cos\sqrt{\frac{g}{l}}\,t\right]. \qquad \text{[C.U.]}$$

27. Two rigidly connected rods make a right angle at their point of junction, and a third rod moves in contact with them, one end sliding on each. If the whole be revolving about one of the first two rods, with angular velocity ω, prove that the kinetic energy of the third rod will be

$$\tfrac{1}{6}ma^2\left\{\left(\frac{d\theta}{dt}\right)^2 + \omega^2 \sin^2\theta\right\},$$

where m is the mass of the rod, a its length, and θ the angle which it makes with the rod about which the system revolves.

Find also an expression for the angular momentum of the third rod about the axis of revolution of the system. [B.E.]

CHAPTER X

VIBRATIONS

82. **Simple Harmonic Motion.**—Simple harmonic motion has been discussed in Chap. VII, pp. 104–116, where it was shown that, denoting the displacement of a particle from its mid-point of travel by x at time t, the acceleration, being proportional to the displacement and directed towards the mid-point of travel, may be written as

$$\frac{d^2x}{dt^2} = -\omega^2 x,$$

or

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad . \qquad . \qquad . \qquad (1).$$

The solution of this equation is

$$x = \mathrm{A}\cos\omega t + \mathrm{B}\sin\omega t \qquad . \qquad . \qquad (2),$$

where the values of the arbitrary constants A and B depend on the conditions in the problem. The solution may also be expressed in either of the forms

$$x = \mathrm{C}\cos(\omega t + \phi) \qquad . \qquad . \qquad . \qquad (3),$$

or

$$x = \mathrm{C}\sin(\omega t + \phi') \qquad . \qquad . \qquad . \qquad (4).$$

The *amplitude* is C, the *periodic time* is $\mathrm{T} = 2\pi/\omega$, and the *frequency* is $f = 1/\mathrm{T} = \omega/2\pi$. Frequency is measured in oscillations, vibrations, or cycles per unit time. A body has made one *oscillation, vibration*, or *cycle* when it has passed once through its whole path and is then about to make another journey, moving in the same direction as at first. For example, if a pendulum swings from rest at A to rest at B and then swings back to rest at A, it has made one oscillation and its path is from A to B and back to A.

It must be understood, however, that the length of the path will gradually decrease when the motion is damped, but then the motion will not be simple harmonic motion.

83. Motion of a Mass Suspended by a Helical Spring.—The analysis which follows may be applied to any system in which the restoring force is proportional to the displacement, the helical spring being merely a particular example.

Consider a mass of weight W suspended by a helical spring, as shown in Fig. 146, where OO is the level of statical equilibrium. Let k be the stiffness or the force which produces unit extension in the spring, then since the extension is proportional to the load, it follows that the statical extension due to the load W is W/k. The mass of the spring will be neglected.

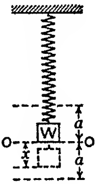

FIG. 146.

Let the mass be pulled down a distance a from the statical level OO and released, then, disregarding resistance due to air, etc., the mass will vibrate with simple harmonic motion and the extremities of its travel will be a distance a below and above the level OO, a being the amplitude.

The case in which the vibrations gradually die away on account of resistance will be considered in the next Art.

Let the displacement of the mass from OO be x at a time t, regarding x as positive when measured below OO, then the additional force extending or compressing the spring is kx, the sign of this force being positive or negative according as x is positive or negative. The corresponding force acting on the mass is $-kx$, and this force always accelerates the mass towards the level OO.

Since mass $\times$ acceleration $=$ accelerating force,

therefore
$$\frac{W}{g}\frac{d^2x}{dt^2} = -kx,$$

which may be written as

$$\frac{d^2x}{dt^2} + \frac{kg}{W}x = 0 \quad . \quad . \quad . \quad (1).$$

Comparing this equation with the general form

$$\frac{d^2x}{dt^2}+\omega^2x=0$$

and its solution $\quad x=\mathrm{A}\cos\omega t+\mathrm{B}\sin\omega t,$

it can be seen that $\quad \omega=\sqrt{\frac{kg}{\mathrm{W}}}$

and that the solution of (1) is

$$x=\mathrm{A}\cos\sqrt{\frac{kg}{\mathrm{W}}}t+\mathrm{B}\sin\sqrt{\frac{kg}{\mathrm{W}}}t \quad . \quad . \quad (2).$$

The arbitrary constants A and B are found by using two conditions. The mass is released and begins to move with zero velocity when $x=a$, therefore, measuring time from this instant, the conditions are $x=a$ and $dx/dt=0$ when $t=0$.

Putting $t=0$ and $x=a$ in (2), it follows that $\mathrm{A}=a$.

Differentiating (2),

$$\frac{dx}{dt}=\sqrt{\frac{kg}{\mathrm{W}}}\left\{-\mathrm{A}\sin\sqrt{\frac{kg}{\mathrm{W}}}t+\mathrm{B}\cos\sqrt{\frac{kg}{\mathrm{W}}}t\right\}.$$

Putting $t=0$ and $dx/dt=0$, it follows that $\mathrm{B}=0$.

Substituting $\mathrm{A}=a$ and $\mathrm{B}=0$ in (2),

then $$x=a\cos\sqrt{\frac{kg}{\mathrm{W}}}t \quad . \quad . \quad . \quad (3).$$

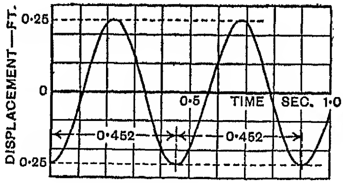

FIG. 147.

As a numerical example, taking $a=0{\cdot}25$ ft., $k=120$ lb. per ft., $\mathrm{W}=20$ lb., and $g=32{\cdot}2$ ft./sec.^{2}, then

$$\omega=\sqrt{\frac{kg}{W}}=\sqrt{\frac{120\times 32\cdot 2}{20}}=13\cdot 90 \text{ rad./sec.},$$

and the displacement at time t is

$$x=0\cdot 25 \cos 13\cdot 90t,$$

the graph of which is shown in Fig. 147.

The periodic time $T=\frac{2\pi}{\omega}=\frac{2\pi}{13\cdot 90}=0\cdot 452$ sec.

The frequency $f=\frac{1}{T}=\frac{13\cdot 90}{2\pi}=2\cdot 21$ cycles/sec.

Since $\frac{1}{\omega}=\sqrt{\frac{W}{kg}}$ and $\frac{W}{k}$ is the statical deflection of the spring due to the load W, the periodic time may also be expressed as

$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{\text{statical deflection}}{g}} \quad . \quad (4).$$

Energy Method of Obtaining the Equation of Motion.—The sum of the kinetic energy and the potential energy of the mass plus the potential energy or strain energy of the spring must be constant. The equation of motion may be obtained by writing down this equality and differentiating with respect to time.

Using the symbols already defined—

Mass—

Kinetic Energy is $\frac{W}{2g}\left(\frac{dx}{dt}\right)^2$.

Potential Energy is $W(a-x)$, taking the lowest position as the datum level.

Spring—

Strain Energy is $\frac{1}{2}$(force $\times$ extension), or

$$\tfrac{1}{2}(W+kx)\left(\frac{W}{k}+x\right)=\frac{1}{2k}(W+kx)^2.$$

Mass and Spring—

Total Energy is $\frac{W}{2g}\left(\frac{dx}{dt}\right)^2+W(a-x)+\frac{1}{2k}(W+kx)^2=\text{const.}$

Differentiating with respect to t,

$$\frac{W}{g}\frac{dx}{dt}\frac{d^2x}{dt^2} - W\frac{dx}{dt} + (W + kx)\frac{dx}{dt} = 0.$$

Therefore $$\frac{d^2x}{dt^2} + \frac{kg}{W}x = 0.$$

This equation may be obtained more quickly if it is noted that the sum of the potential and strain energies is

$$W(a - x) + \frac{1}{2k}(W + kx)^2 = Wa - Wx + \frac{W^2}{2k} + Wx + \tfrac{1}{2}kx^2$$

$$= \tfrac{1}{2}kx^2 + \text{constant terms}$$

and $\frac{1}{2}kx^2$ is the additional strain energy stored in the spring due to vibration.

Therefore it follows that the kinetic energy of the mass plus the additional strain energy stored in the spring due to vibration is constant, or

$$\frac{W}{2g}\left(\frac{dx}{dt}\right)^2 + \tfrac{1}{2}kx^2 = \text{const.}$$

Differentiating with respect to t,

$$\frac{W}{g}\frac{dx}{dt}\frac{d^2x}{dt^2} + kx\frac{dx}{dt} = 0,$$

from which, as before,

$$\frac{d^2x}{dt^2} + \frac{kg}{W}x = 0.$$

84. Damped Vibrations.—Suppose the motion of a vibrating mass is opposed by a force which is proportional to the velocity, and let this force be denoted by $-b\frac{dx}{dt}$, where b is a constant and the minus sign indicates that the force opposes the motion.

Using the notation of the preceding Art., the equation

$$\text{mass} \times \text{acceleration} = \text{accelerating force}$$

becomes $$\frac{W}{g}\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx,$$

from which $$\frac{d^2x}{dt^2}+\frac{bg}{W}\frac{dx}{dt}+\frac{kg}{W}x=0,$$

or, for convenience, putting $\frac{bg}{W}=2c$ and $\frac{kg}{W}=\omega^2$,

$$\frac{d^2x}{dt^2}+2c\frac{dx}{dt}+\omega^2x=0 \quad . \quad . \quad . \quad (1),$$

which represents the damped vibrations such as would occur when a mass oscillates at the end of a spring, and the amplitude of the motion gradually decreases on account of air resistance, etc., until finally the mass comes to rest. The time which elapses before the motion ceases depends on the value of the damping constant c, and if c is of sufficient magnitude the mass will not oscillate, but will return towards the level of statical equilibrium without quite reaching it.

The mass will oscillate provided c is less than ω. In this case the solution of (1) is *

$$x=e^{-ct}(\mathrm{A}\cos\sqrt{\omega^2-c^2}\,t+\mathrm{B}\sin\sqrt{\omega^2-c^2}\,t) \quad . \quad (2),$$

which can also be written in the form

$$x=\mathrm{C}e^{-ct}\cos(\sqrt{\omega^2-c^2}\,t-\phi) \quad . \quad . \quad (3),$$

where C and ϕ are arbitrary constants.

To determine C and ϕ, suppose $t=0$ and $\frac{dx}{dt}=0$ when $x=a$, where a is the maximum displacement (Fig. 146).

Putting $t=0$ and $x=a$ in (3),

$$a=\mathrm{C}\cos(-\phi)=\mathrm{C}\cos\phi,$$

from which $$\mathrm{C}=\frac{a}{\cos\phi}.$$

Differentiating (3),

$$\frac{dx}{dt}=\mathrm{C}e^{-ct}\{-\sqrt{\omega^2-c^2}\sin(\sqrt{\omega^2-c^2}\,t-\phi)\}$$
$$-\mathrm{C}ce^{-ct}\cos(\sqrt{\omega^2-c^2}\,t-\phi).$$

* The method of solving equations of this type is explained in the author's *Mathematics*, Longmans, Green & Co., or in any book on Differential Equations.

Putting $t=0$ and $\dfrac{dx}{dt}=0$, and remembering that

$$\sin(-\phi)=-\sin\phi \quad \text{and} \quad \cos(-\phi)=\cos\phi,$$

$$0=C\sqrt{\omega^2-c^2}\sin\phi - Cc\cos\phi,$$

from which

$$\tan\phi=\frac{c}{\sqrt{\omega^2-c^2}} \quad \text{or} \quad \phi=\tan^{-1}\frac{c}{\sqrt{\omega^2-c^2}}.$$

The angle ϕ can be represented as shown in Fig. 148. Since $C=a/\cos\phi$, taking the value of $\cos\phi$ from the triangle it can be seen that $C=\omega a/\sqrt{\omega^2-c^2}$.

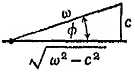

FIG. 148.

Equation (3) may now be written as

$$x=\frac{\omega a}{\sqrt{\omega^2-c^2}}e^{-ct}\cos(\sqrt{\omega^2-c^2}\,t-\phi) \quad . \quad (4),$$

where
$$\phi=\tan^{-1}\frac{c}{\sqrt{\omega^2-c^2}}.$$

As a numerical example, taking $a=0{\cdot}25$ ft., $k=120$ lb. per ft., $W=20$ lb., and $g=32{\cdot}2$ ft./sec.^{2}, then

$$\omega=\sqrt{\frac{kg}{W}}=\sqrt{\frac{120\times 32{\cdot}2}{20}}=13{\cdot}90 \text{ rad./sec.},$$

as in the preceding Art.

Taking the damping force $b\dfrac{dx}{dt}=2\dfrac{dx}{dt}$ lb., the velocity being in ft./sec., then $2c=\dfrac{bg}{W}=\dfrac{2\times 32{\cdot}2}{20}$ and $c=1{\cdot}61$ rad./sec.

Therefore

$$\sqrt{\omega^2-c^2}=\sqrt{13{\cdot}90^2-1{\cdot}61^2}=13{\cdot}81 \text{ rad./sec.}$$

and
$$\frac{\omega a}{\sqrt{\omega^2-c^2}}=\frac{13{\cdot}90\times 0{\cdot}25}{13{\cdot}81}=0{\cdot}252 \text{ ft.};$$

also $\tan\phi=\dfrac{c}{\sqrt{\omega^2-c^2}}=\dfrac{1{\cdot}61}{13{\cdot}81}=0{\cdot}117$, from which $\phi=6°\,40'$.

The value of $\sqrt{\omega^2 - c^2}$ becomes $13{\cdot}81 \times \frac{180}{\pi} = 791$ when expressed in degrees/sec.

Substituting numerical values in (4),

$$x = 0{\cdot}252e^{-1{\cdot}61t} \cos (791t - 6^\circ\ 40') \quad . \quad (5),$$

the graph of which is shown by the thick line curve in Fig. 149.

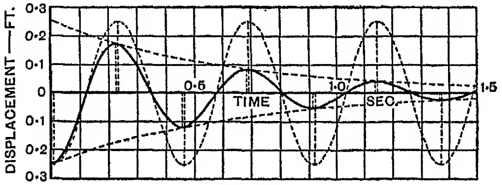

FIG. 149.

The graphs of $0{\cdot}252 \cos (791t - 6^\circ\ 40')$ and $\pm\, 0{\cdot}252e^{-1{\cdot}61t}$ are also given, and it can be seen how the damping factor $e^{-1{\cdot}61t}$ quickly damps out the motion. It is of interest to notice that the highest and lowest points (*turning-points*) on the damped curve occur slightly before the highest and lowest points on the cosine curve. This shows that the mass takes less time to move from the level of statical equilibrium to an extreme position than it takes to make the return journey.

The periodic time is $T = \frac{2\pi}{\sqrt{\omega^2 - c^2}} = \frac{2\pi}{13{\cdot}81} = 0{\cdot}455$ sec.

When there is no damping, $c = 0$, and the ratio

$$\frac{\text{Periodic time with damping}}{\text{Periodic time without damping}} = \frac{\omega}{\sqrt{\omega^2 - c^2}} = \frac{13{\cdot}90}{13{\cdot}81} = 1{\cdot}007,$$

which is greater than unity, showing that damping increases the periodic time.

The displacements at successive turning-points will now be examined. It has been stated that the displacement x

at any time t is obtained from equation (3),

$$x = Ce^{-ct}\cos(\sqrt{\omega^2 - c^2}\,t - \phi).$$

Differentiating,

$$\frac{dx}{dt} = Ce^{-ct}\{-\sqrt{\omega^2 - c^2}\sin(\sqrt{\omega^2 - c^2}\,t - \phi)\}$$
$$- Cce^{-ct}\cos(\sqrt{\omega^2 - c^2}\,t - \phi).$$

Now $\frac{dx}{dt} = 0$ at turning-points, therefore by equating to zero it follows that

$$\tan(\sqrt{\omega^2 - c^2}\,t - \phi) = -\frac{c}{\sqrt{\omega^2 - c^2}},$$

or $$\sqrt{\omega^2 - c^2}\,t - \phi = \tan^{-1}\left\{-\frac{c}{\sqrt{\omega^2 - c^2}}\right\}.$$

There are many angles θ_1, θ_2, etc., at intervals of π radians whose tangent is $-c/\sqrt{\omega^2 - c^2}$, therefore, denoting the successive turning-points by the suffixes, 1, 2, etc.,

$$\sqrt{\omega^2 - c^2}\,t_1 - \phi = \theta_1$$

and $$\sqrt{\omega^2 - c^2}\,t_2 - \phi = \theta_2 = \theta_1 + \pi.$$

From these equations,

$$t_2 - t_1 = \frac{\pi}{\sqrt{\omega^2 - c^2}}.$$

Therefore $$\frac{x_2}{x_1} = \frac{e^{-ct_2}\cos(\sqrt{\omega^2 - c^2}\,t_2 - \phi)}{e^{-ct_1}\cos(\sqrt{\omega^2 - c^2}\,t_1 - \phi)}$$
$$= \frac{e^{-ct_2}\cos(\theta_1 + \pi)}{e^{-ct_1}\cos\theta_1}$$
$$= -e^{-c(t_2 - t_1)} = -e^{-c\pi/\sqrt{\omega^2 - c^2}}$$

or, in general terms,

$$\frac{x_n}{x_{n-1}} = -e^{-c\pi/\sqrt{\omega^2 - c^2}}$$

the ratio being negative because the cosines have the same numerical value but differ in sign. The values x_1, x_2, etc.,

form a series in geometrical progression and the common ratio is $-e^{-c\pi/\sqrt{\omega^2-c^2}}$. If x_1 is positive, then x_2 is negative and x_3 is positive, and so on.

Changing the sign of the common ratio,

$$-\frac{x_n}{x_{n-1}}=e^{-c\pi/\sqrt{\omega^2-c^2}}$$

then it is evident that $-c\pi/\sqrt{\omega^2-c^2}$ is the logarithm to the base e of the positive ratio $-x_n/x_{n-1}$. The quantity $c\pi/\sqrt{\omega^2-c^2}$ is known as the *logarithmic decrement.*

85. Forced and Damped Vibrations.—A vibrating mass whose motion is forced as well as damped will now be examined. Consider the helical spring carrying a mass of weight W as in Art. 83, but let the top of the spring be given simple harmonic motion with a vertical amplitude a_1 as indicated in Fig. 150, where O_1O_1 is the mean position of the top of the spring and OO is the level of the mass when in statical equilibrium. The amplitude of the motion of the mass is labelled a, but this value is as yet unknown.

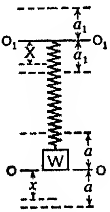

FIG. 150.

Suppose the mass descends a distance x when the top of the spring descends a distance X. If x is greater than X, then the extension of the spring is $x-X$, and if k is the force per unit extension, the corresponding force acting upwards on the mass is $k(x-X)$. Since the positive direction is downwards, this force may be expressed as $-k(x-X)$.

Taking $-b\dfrac{dx}{dt}$ as the damping force, as in the preceding Art., the equation of motion becomes

$$\frac{W}{g}\frac{d^2x}{dt^2}=-b\frac{dx}{dt}-k(x-X) \quad . \quad . \quad (1),$$

or, dividing by $\dfrac{W}{g}$ and for convenience putting $\dfrac{bg}{W}=2c$ and

$\frac{kg}{W} = \omega^2$, the equation may be written in the form

$$\frac{d^2x}{dt^2} + 2c\frac{dx}{dt} + \omega^2 x = \omega^2 X.$$

Since the top of the spring moves with simple harmonic motion with an amplitude a_1, let $X = a_1 \sin pt$ as indicated in Fig. 151, then the equation of motion becomes

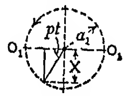

FIG. 151.

$$\frac{d^2x}{dt^2} + 2c\frac{dx}{dt} + \omega^2 x = \omega^2 a_1 \sin pt \quad . \quad (2).$$

The complete solution of this equation * is the sum of two solutions called the *complementary function* and the *particular integral.* The complementary function is obtained by equating the left-hand side of the equation to zero and it is the solution given in equation (3), p. 167, that is

$$x = Ce^{-ct} \cos (\sqrt{\omega^2 - c^2}\, t - \phi) \quad . \quad . \quad (3).$$

The particular integral is a particular solution which satisfies the whole of equation (2) and it can be shown to be

$$x = \frac{\omega^2 a_1}{\sqrt{(\omega^2 - p^2)^2 + 4c^2p^2}} \sin (pt - \epsilon) \quad . \quad (4),$$

where $\epsilon = \tan^{-1} \frac{2cp}{\omega^2 - p^2}$. The angle ϵ is the phase difference † between the motion of the mass and the motion of the top of the spring, the mass lagging by this angle.

The complete solution of (2) is the sum of (3) and (4), but the damped natural vibrations represented by (3) eventually die away and after a time the motion of the mass is represented by (4).

The amplitude of the motion of the mass is therefore

$$\frac{\omega^2 a_1}{\sqrt{(\omega^2 - p^2)^2 + 4c^2p^2}},$$

* [illegible] *ematics*, Longmans, Green

† For definition see p. 167.

which was denoted by a in Fig. 150, and the forced amplitude of the top of the spring is a_1.

Therefore the ratio

$$\frac{\text{Amplitude of mass}}{\text{Amplitude of top of spring}} = \frac{\omega^2 a_1}{\sqrt{(\omega^2 - p^2)^2 + 4c^2p^2}} \cdot \frac{1}{a_1},$$

which may be called the *magnification of amplitude* or *magnification factor* and written as

$$\text{Magnification of amplitude} = \frac{1}{\sqrt{\left(1 - \frac{p^2}{\omega^2}\right)^2 + \left(\frac{2c}{\omega}\right)^2 \frac{p^2}{\omega^2}}}.$$

This ratio is plotted (Fig. 152) against the ratio of the applied frequency $p/2\pi$ to the natural frequency $\omega/2\pi$,

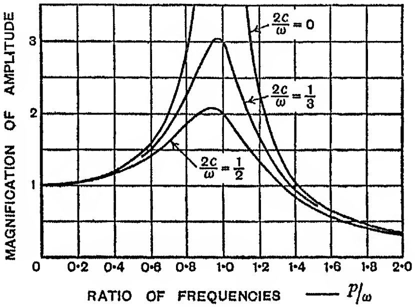

Fig. 152.

—that is, the ratio p/ω—for three values of $2c/\omega$. Since c is dependent on the damping force, a comparison of the curves shows how the ratio of the amplitudes is affected by the damping; the greater the damping, the less the magnification of the amplitude. The curves also show that when the ratio p/ω reaches a certain value, which is different for each curve, the magnification of the amplitude becomes less than unity—that is to say, the amplitude is actually reduced.

If there is no damping, $c = 0$, then the magnification

becomes infinite when $p=\omega$—that is, when the applied and natural frequencies are equal. This equality of the frequencies is called *resonance*. In practice there is always some damping, so that c is not zero although it may be very small. It can be seen from the two lower curves that when the applied and natural frequencies are equal, or nearly equal, the amplitude of the motion of the mass may become large, but it will not be infinite. It should be noticed that when there is damping, then the maximum amplitude occurs when p is less than ω.

To understand how the relation between p and ω affects the phase difference $\epsilon=\tan^{-1}\dfrac{2cp}{\omega^2-p^2}$ and also to understand resonance, it is worth making a simple experiment. Secure a mass to one end of a piece of elastic or a light helical spring and hold the other end in the hand.

(*a*) Move the hand up and down very slowly and the mass will follow the motion of the hand.

(*b*) Move the hand up and down rapidly and it will be found that the mass goes down when the hand goes up and *vice versa*. If the hand is moved fast enough the mass will be almost motionless.

(*c*) Move the hand at some intermediate speed, found by trial, and the amplitude of the motion of the mass will become large. This is what happens at resonance.

Now consider these effects from a mathematical standpoint. The phase difference is $\epsilon=\tan^{-1}\dfrac{2cp}{\omega^2-p^2}$. When $p=\omega$, $\epsilon=\tan^{-1}\infty=\pi/2$, and this is the value of ϵ at resonance. When p is less than ω, ϵ is less than $\pi/2$ (Fig. 153, top). If p is nearly zero, then ϵ is nearly zero, and this is what happened in experiment (*a*) when the hand and the mass moved together. When p is greater than ω, ϵ is greater than $\pi/2$ (Fig. 153, bottom) because ω^2-p^2 is negative, and the greater p is compared with ω the closer ϵ approaches to the value π. This is what happened

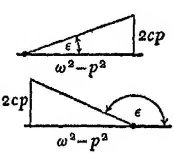

FIG. 153.

when the hand and the mass moved in opposite directions in experiment (*b*).

In the preceding analysis a spring loaded at one end was given a known *periodic displacement* at the other end. The equations also apply when a system is disturbed by a known *periodic force*. For example, consider an engine, of weight W, with an out-of-balance flywheel and supported on a spring as shown diagrammatically in Fig. 154.

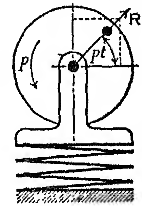

FIG. 154.

Let the vertical periodic disturbing force due to the out-of-balance flywheel be R sin *pt*, where R is a variable force depending on p^2, p being the speed of rotation in radians per second, then equation (1) becomes

$$\frac{W}{g}\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + R\sin pt,$$

the upward direction being taken as positive, and this equation reduces to

$$\frac{d^2x}{dt^2} + 2c\frac{dx}{dt} + \omega^2 x = \frac{Rg}{W}\sin pt,$$

using the same notation as before.

Now suppose that R, the maximum value of the disturbing force at a particular speed p, would give the spring a deflection a_1 if applied statically, then

$$R = ka_1 \quad \text{and} \quad \frac{Rg}{W} = \frac{kg}{W}a_1 = \omega^2 a_1,$$

therefore
$$\frac{d^2x}{dt^2} + 2c\frac{dx}{dt} + \omega^2 x = \omega^2 a_1 \sin pt,$$

which is the same as equation (2), p. 172.

Since R is a variable depending on p^2, it must be remembered that the statical deflection a_1 is also a variable.

The magnification factor can now be interpreted as the ratio

$$\frac{\text{Amplitude of the mass}}{\text{Deflection due to a statical force R}}.$$

86. Torsional Vibrations—Shaft Carrying One Flywheel.—A shaft held at one end and carrying a flywheel at the other end is shown diagrammatically in Fig. 155. If the flywheel is turned through a small angle θ about OX, the axis of the shaft, and then released, it will oscillate with simple harmonic motion, for the torque in the shaft is proportional to the angle of twist. It will be assumed that the mass of the shaft is negligible compared with the mass of the flywheel, and damping will be neglected. Periodic motion may be superimposed on the system when it is driven at the end O instead of being fixed. In practice, the object is to reduce the oscillations as much as possible.

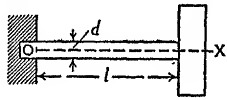

FIG. 155.

Considering the shaft, let l be the length, d the diameter, θ the angle of twist caused by a torque T_q at any time t, then if C is the modulus of rigidity it is known that $T_q = \frac{C\theta}{l}\frac{\pi d^4}{32}$. This torque opposes the motion of the flywheel, and therefore the torque on the flywheel is

$$T_q = -\frac{C\theta}{l}\frac{\pi d^4}{32} \quad . \quad . \quad . \quad (1),$$

the sign being negative because the torque acts towards the unstrained position—that is, when θ is positive the torque is negative and *vice versa*.

Considering the motion of the flywheel,

$$T_q = I\frac{d^2\theta}{dt^2} \quad . \quad . \quad . \quad (2),$$

where I is the moment of inertia of the flywheel about the axis of rotation.

From (1) and (2), $I\frac{d^2\theta}{dt^2} = -\frac{C\theta}{l}\frac{\pi d^4}{32}$,

or

$$\frac{d^2\theta}{dt^2} + \frac{C\pi d^4}{32Il}\theta = 0 \quad . \quad . \quad . \quad (3).$$

Writing ω^2 for $C\pi d^4/32Il$, then

$$\frac{d^2\theta}{dt^2}+\omega^2\theta=0 \quad . \quad . \quad . \quad (4),$$

which is the equation of simple harmonic motion, and the general solution is

$$\theta=\mathrm{A}\cos\omega t+\mathrm{B}\sin\omega t \quad . \quad . \quad (5).$$

The periodic time is

$$\mathrm{T}=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{32Il}{C\pi d^4}} \quad . \quad . \quad . \quad (6).$$

This formula may be expressed in another way. Since the torque in the shaft is $C\theta\pi d^4/32l$, therefore, putting $\theta=1$,

$$\text{Torque per unit angle of twist}=C\pi d^4/32l.$$

Therefore

$$\mathrm{T}=2\pi\sqrt{\frac{\text{Moment of inertia of flywheel about axis}}{\text{Torque per unit angle of twist}}} \quad . \quad (7).$$

Example.—To find the periodic time from the following data—Weight of flywheel, $W=100$ lb.; radius of gyration of flywheel, $k=8$ in.; $g=32{\cdot}2$ ft./sec.^{2}; $C=11{\cdot}7\times10^6$ lb./in.^{2}; $l=10$ in.; and $d=1{\cdot}5$ in.

The calculations will be made in pound, inch, and second units.

$$\mathrm{I}=\frac{W}{g}k^2=\frac{100}{32{\cdot}2\times12}\times8^2=16{\cdot}56\,\frac{\text{lb. in.}^2}{\text{in./sec.}^2}.$$

$$\mathrm{T}=2\pi\sqrt{\frac{32Il}{C\pi d^4}}=2\pi\sqrt{\frac{32\times16{\cdot}56\times10}{11{\cdot}7\times10^6\times\pi\times1{\cdot}5^4}}=0{\cdot}0335\text{ sec.}$$

Checking the units:

$$\sqrt{\frac{\text{lb. in.}^2}{\text{in./sec.}^2}\times\frac{1}{\text{lb./in.}^2}\times\frac{\text{in.}}{\text{in.}^4}}=\text{sec.}$$

87. Torsional Vibrations—Shaft Diameter Not Uniform. —The results obtained in the preceding Art. require modification if the diameter of the shaft is not uniform.

As before, the shaft is held at one end and carries a flywheel at the other end, but it will be assumed now that the total length is l_1+l_2, the length l_1 having a diameter d_1 and the length l_2 having a diameter d_2, as shown in Fig. 156.

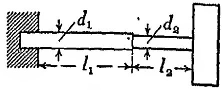

FIG. 156.

Let θ_1 and θ_2 be the angles of twist in the lengths l_1 and l_2 respectively, then the total twist is $\theta=\theta_1+\theta_2$.

The torque is the same at every section of the shaft and denoting the torque on the flywheel by T_q,

$$T_q=-\frac{C\theta_1}{l_1}\frac{\pi d_1^4}{32}=-\frac{C\theta_2}{l_2}\frac{\pi d_2^4}{32},$$

from which

$$\theta_1=-T_q\frac{32}{C\pi}\frac{l_1}{d_1^4} \quad \text{and} \quad \theta_2=-T_q\frac{32}{C\pi}\frac{l_2}{d_2^4},$$

then
$$\theta=\theta_1+\theta_2=-T_q\frac{32}{C\pi}\left(\frac{l_1}{d_1^4}+\frac{l_2}{d_2^4}\right),$$

or
$$T_q=-\frac{C\pi\theta}{32\left(\frac{l_1}{d_1^4}+\frac{l_2}{d_2^4}\right)} \quad . \quad . \quad (1).$$

Also
$$T_q=I\frac{d^2\theta}{dt^2} \quad . \quad . \quad . \quad (2).$$

From (1) and (2)

$$\frac{d^2\theta}{dt^2}+\frac{C\pi\theta}{32I\left(\frac{l_1}{d_1^4}+\frac{l_2}{d_2^4}\right)}=0 \quad . \quad . \quad . \quad (3),$$

and the periodic time is

$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{32I}{C\pi}\left(\frac{l_1}{d_1^4}+\frac{l_2}{d_2^4}\right)} \quad . \quad . \quad (4).$$

Suppose the shaft is made up of lengths l_1, l_2, l_3, etc., having diameters d_1, d_2, d_3, etc., then, denoting

$$\left(\frac{l_1}{d_1^4}+\frac{l_2}{d_2^4}+\frac{l_3}{d_3^4}+\ .\ .\ .\right) \quad \text{by} \quad \sum\frac{l}{d^4},$$

the periodic time becomes

$$T=2\pi\sqrt{\frac{32I}{C\pi}\sum\frac{l}{d^4}} \quad . \quad . \quad . \quad (5).$$

88. Torsional Vibrations—Shaft Carrying Two Flywheels.—A shaft of length l and diameter d carries flywheels 1 and 2, one at each end as shown in Fig. 157. It is required to find the periodic time of the torsional vibrations when the flywheels are oscillating in opposite directions.

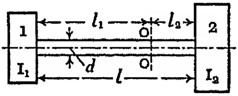

Fig. 157.

Since the flywheels are oscillating in opposite directions, there will be no movement at some section OO and the periodic times of the flywheels will be equal. The centre of the section OO is called a *node*, or the section may be called a *nodal section.* Once the position of OO is known, then either flywheel with its part of the shaft up to OO may be treated as shown in Art. 86.

Let the section OO be l_1 from flywheel 1 and l_2 from flywheel 2, then $l_1+l_2=l$. Let I_1 and I_2 be the moments of inertia of the flywheels about the axis of rotation and let T_1 and T_2 be their periodic times, which, as already stated, are equal.

It was shown in Art. 86 that the periodic time $T=2\pi\sqrt{32Il/C\pi d^4}$, and this formula may be applied to each part of the system.

Therefore $$T_1=2\pi\sqrt{\frac{32I_1l_1}{C\pi d^4}} \quad . \quad . \quad . \quad (1),$$

and $$T_2=2\pi\sqrt{\frac{32I_2l_2}{C\pi d^4}} \quad . \quad . \quad . \quad (2).$$

But $T_1=T_2$, therefore

$$I_1l_1=I_2l_2 \quad . \quad . \quad . \quad (3).$$

Also $$l_1+l_2=l \quad . \quad . \quad . \quad (4).$$

Solving the simultaneous equations (3) and (4) gives

$$l_1=\frac{I_2l}{I_1+I_2} \quad \text{and} \quad l_2=\frac{I_1l}{I_1+I_2}.$$

Substituting the value of l_1 in (1) or the value of l_2 in (2),

$$\text{Periodic time } T_1 = T_2 = 2\pi\sqrt{\frac{32 I_1 I_2 l}{C\pi d^4 (I_1 + I_2)}} \quad . \quad (5).$$

Example.—The shaft shown in Fig. 158 carries two heavy masses at A and B and is driven by a light gear situated at CC. The weight of A is 800 lb. and its radius of gyration is 27 in.; the corresponding values for B are 1200 lb. and 33 in. The shaft diameter between CC and B, marked X in., is undecided. Assuming it to be $3\frac{1}{2}$ in., determine the frequency of free torsional oscillations of the system. Thereafter, determine what X should be if the node of the vibration is to be in the plane, CC, of the drive. Deduce any formula used.

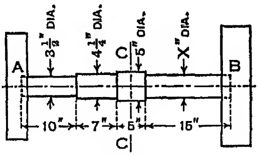

Fig. 158.

Rigidity Modulus $= 12 \times 10^6$ lb. per sq. in. [U.L.]

Taking $X = 3\frac{1}{2}$ in., the first step is to find the position of the node of the vibration. Assuming that the node is in the 15-inch length on the right, let it be at a distance x from the end B of the shaft. Let I_A and I_B be the moments of inertia of the masses about the axis of rotation, and let the mass of the shaft be neglected.

If I_A were equal to I_B and if the shaft were of uniform diameter, the node would be midway between the masses, or 18·5 inches from the end B. But $I_B = \frac{1200}{g} \times 33^2$ being considerably greater than $I_A = \frac{800}{g} \times 27^2$ causes the node to be nearer B than A, and the greater diameters on the 7-inch and 5-inch lengths have a similar effect.

Therefore it seems highly probable that x is less than 15 inches, as already assumed.

$$\text{Periodic time of mass B is } T_B = 2\pi\sqrt{\frac{32 I_B x}{C\pi X^4}} \qquad (1).$$

From equation (5) of the preceding Art., it follows that

$$\text{Periodic time of mass A is } T_A = 2\pi\sqrt{\frac{32 I_A}{C\pi}\sum\frac{l}{d^4}} \quad (2).$$

But $\qquad T_B = T_A,\quad$ therefore $\quad I_B\frac{x}{X^4} = I_A\sum\frac{l}{d^4},$

or, substituting numerical values, cancelling g,

$$1200 \times 33^2\frac{x}{3{\cdot}5^4} = 800 \times 27^2\left(\frac{10}{3{\cdot}5^4} + \frac{7}{4{\cdot}25^4} + \frac{5}{5^4} + \frac{15 - x}{3{\cdot}5^4}\right).$$

Multiplying each side by $\dfrac{3{\cdot}5^4}{1200 \times 33^2}$ and simplifying,

$$x = \tfrac{54}{121}(10 + 3{\cdot}22 + 1{\cdot}20 + 15 - x)$$

from which $x = 9{\cdot}08$ inches, and the assumption that this value would be less than 15 inches is correct.

Frequency $\qquad f = \dfrac{1}{T_B} = \dfrac{1}{2\pi}\sqrt{\dfrac{C\pi X^4}{32 I_B x}}.$

Substituting numerical values and taking $g = 32{\cdot}2 \times 12$ in./sec.^{2},

$$f = \frac{1}{2\pi}\sqrt{\frac{12 \times 10^6 \times \pi \times 3{\cdot}5^4 \times 32{\cdot}2 \times 12}{32 \times 1200 \times 33^2 \times 9{\cdot}08}} = 12{\cdot}1 \text{ oscillations/sec.}$$

The value of X will now be found so that the node of the vibration may be in the plane CC.

The periodic times of the masses A and B are equal, therefore

$$I_A\sum\frac{l}{d^4}\text{ (from A to CC)} = I_B\sum\frac{l}{d^4}\text{ (from CC to B)}.$$

Substituting numerical values, cancelling g,

$$800 \times 27^2\left(\frac{10}{3{\cdot}5^4} + \frac{7}{4{\cdot}25^4} + \frac{2{\cdot}5}{5^4}\right) = 1200 \times 33^2\left(\frac{2{\cdot}5}{5^4} + \frac{15}{X^4}\right).$$

Simplifying and solving for X, gives $X = 4{\cdot}48$ inches.

89. Torsional Vibrations—Unloaded Shaft Fixed at One End.—Consider a shaft OX (Fig. 159), of diameter d and

length l, fixed at O and making torsional vibrations about its longitudinal axis. Let P be a section at a distance x from O, and let Q be a section near to P and at a distance $x+\delta x$ from O.

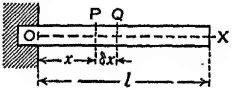

FIG. 159.

When a uniform shaft of length l is twisted through an angle θ by a torque T_q, then $T_q=I_pC\theta/l$, where $I_p=\pi d^4/32$ is the polar moment of inertia, or second moment of area about a perpendicular axis through the centre, of a cross-section and C is the modulus of rigidity, and the torque has the same value at every section.

In the case to be considered the torque varies from section to section on account of the inertia of the vibrating shaft. Let $\delta\theta$ be the angle of twist in the length δx, then $T_q=I_pC\delta\theta/\delta x$, and if δx is made indefinitely small the torque at the section P is ultimately obtained.

Torque at P is $$T_q=I_pC\frac{\partial\theta}{\partial x} \quad . \quad . \quad . \quad (1),$$

the symbols of partial differentiation being used because time t is also a variable.

Torque at Q is

$$T_q+\frac{\partial T_q}{\partial x}\delta x=I_pC\frac{\partial\theta}{\partial x}+I_pC\frac{\partial^2\theta}{\partial x^2}\delta x \quad . \quad . \quad (2),$$

the terms on the right of the equality sign being obtained from (1).

From (1) and (2), by subtraction, the change of torque in the length δx is

$$I_pC\frac{\partial^2\theta}{\partial x^2}\delta x \quad . \quad . \quad . \quad (3).$$

The change of torque given by (3) produces angular acceleration in the mass of length δx and is equal to $I\partial^2\theta/\partial t^2$ or

$$\frac{\rho I_p\delta x}{g}\frac{\partial^2\theta}{\partial t^2} \quad . \quad . \quad . \quad (4),$$

where ρ is the density and $I = \rho I_p \delta x/g$ is the moment of inertia of the mass about the axis of rotation.

Equating (4) to (3) and simplifying, then

$$\frac{\partial^2\theta}{\partial t^2} = a^2\frac{\partial^2\theta}{\partial x^2} \quad . \quad . \quad . \quad (5),$$

where the substitution $a^2 = Cg/\rho$ is made for convenience.

Suppose that the free end of the shaft is twisted through an angle θ_0 and released, then, measuring time from the instant the shaft is released and assuming that the vibrations are harmonic at every section,

let $$\theta = X \cos \omega t \quad . \quad . \quad . \quad (6),$$

where X is a function of x only.

From (6), by partial differentiation,

$$\frac{\partial^2\theta}{\partial t^2} = -\omega^2 X \cos \omega t = -\omega^2\theta,$$

and $$\frac{\partial^2\theta}{\partial x^2} = \frac{d^2X}{dx^2}\cos \omega t = \frac{d^2X}{dx^2}\frac{\theta}{X}.$$

Substituting these values in (5),

$$-\omega^2\theta = a^2\frac{d^2X}{dx^2}\frac{\theta}{X},$$

or $$\frac{d^2X}{dx^2} + \frac{\omega^2}{a^2}X = 0 \quad . \quad . \quad . \quad (7).$$

The solution of this equation is

$$X = A \cos \frac{\omega}{a}x + B \sin \frac{\omega}{a}x,$$

and substituting this value of X in (6), then

$$\theta = \left(A \cos \frac{\omega}{a}x + B \sin \frac{\omega}{a}x\right) \cos \omega t \quad . \quad (8)$$

is the solution of (5).

The values of the arbitrary constants A and B will now be determined.

At the fixed end, $x = 0$ and $\theta = 0$ for all values of t,

therefore $0 = (A + 0) \cos \omega t$, from which $A = 0$.

Putting $A=0$ in (8),

$$\theta = B \sin \frac{\omega}{a}x \cos \omega t \quad . \quad . \quad . \quad (9).$$

At the free end,

$$x=l \quad \text{and} \quad \theta=\theta_0 \quad \text{when} \quad t=0,$$

therefore

$$\theta_0 = B \sin \frac{\omega}{a}l, \quad \text{from which} \quad B=\theta_0/\sin \frac{\omega}{a}l.$$

Substituting this value of B in (9), then

$$\theta = \theta_0 \frac{\sin \frac{\omega}{a}x}{\sin \frac{\omega}{a}l} \cos \omega t \quad . \quad . \quad . \quad (10).$$

Differentiating, $\dfrac{\partial \theta}{\partial x} = \dfrac{\omega}{a}\theta_0 \dfrac{\cos \frac{\omega}{a}x}{\sin \frac{\omega}{a}l} \cos \omega t.$

Now at the free end of the shaft the torque is always zero, so that when $x=l$, $\partial\theta/\partial x=0$ for all values of t, therefore

$$0 = \frac{\omega}{a}\theta_0 \cot \frac{\omega}{a}l \cos \omega t, \quad \text{from which} \quad \cot \frac{\omega}{a}l = 0,$$

and so $\qquad \dfrac{\omega}{a}l = \dfrac{\pi}{2}, \quad 3\dfrac{\pi}{2}, \quad 5\dfrac{\pi}{2}, \quad$ etc.,

and the fundamental vibration is given by putting $\dfrac{\omega}{a}l = \dfrac{\pi}{2}$, or $\sin \dfrac{\omega}{a}l = 1$, in (10).

Therefore $\qquad \theta = \theta_0 \sin \dfrac{\omega}{a}x \cos \omega t,$

or

$$\theta = \theta_0 \sin \frac{\pi x}{2l} \cos \frac{\pi a}{2l}t \quad . \quad . \quad . \quad (11).$$

The periodic time of this vibration is

$$T = 2\pi \Big/ \frac{\pi a}{2l} = \frac{4l}{a} = 4l\sqrt{\frac{\rho}{Cg}}.$$

The frequency is $f=\frac{1}{T}=\frac{1}{4l}\sqrt{\frac{Cg}{\rho}}$.

As a numerical example suppose the shaft is 5 ft. long, and take $C=11\cdot7\times10^6$ lb./in.^{2}, $\rho=490$ lb./ft.^{3} (for steel), and $g=32\cdot2$ ft./sec.^{2}.

Working in lb., ft., and sec. units, then

$$f=\frac{1}{4\times5}\sqrt{\frac{11\cdot7\times10^6\times144\times32\cdot2}{490}}=526 \text{ oscillations/sec.},$$

which is seen to be very high. It will be noticed that the diameter of the shaft does *not* enter into the calculations, and provided the cross-section is circular, the shaft may be solid or hollow.

90. Transverse Vibrations of a Beam Carrying One Concentrated Load.—If a beam is deflected and suddenly released it will make transverse vibrations. The simplest case occurs when the beam carries a concentrated load of such magnitude that the mass of the beam may be neglected. A particular example is illustrated in Fig. 160. The deflection of the beam is proportional to the load W, and if the beam is deflected beyond the position of statical equilibrium and released, then, disregarding damping, the load will vibrate with simple harmonic motion, as did the load supported by a helical spring (Art. 83, p. 163). Since the beam acts as a spring the mathematics is the same in each case.

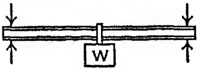

FIG. 160.

$$\text{Periodic time } T=2\pi\sqrt{\frac{\delta}{g}},$$

where δ is the statical deflection due to the load W.

$$\text{Frequency } f=\frac{1}{T}=\frac{1}{2\pi}\sqrt{\frac{g}{\delta}}.$$

The formulæ are not strictly correct if, as the beam vibrates, the load turns about an axis perpendicular to the plane of vibration, as shown diagrammatically in

Fig. 161, where the two short lines, one continuous and one dotted, indicate two positions of the load. However, the angular motion and the rotational energy involved are usually small and may be neglected.

Fig. 161.

Example.—To find the natural frequency of vibration when a load of 1500 lb. is carried at the centre of a beam, 20 ft. long, simply supported at each end. The statical deflection for a central load W is $Wl^3/48EI$ and it is given that $I=122{\cdot}3$ in.^{4} and $E=30\times10^6$ lb./in.^{2}.

$$\text{Frequency } f=\frac{1}{2\pi}\sqrt{\frac{g}{\delta}}=\frac{1}{2\pi}\sqrt{\frac{48EIg}{Wl^3}}.$$

Using lb., inch, and sec. units,

$$f=\frac{1}{2\pi}\sqrt{\frac{48\times30\times10^6\times122{\cdot}3\times32{\cdot}2\times12}{1500\times20^3\times12^3}}$$

$$=9{\cdot}12 \text{ oscillations/sec.}$$

91. Transverse Vibrations of a Beam Carrying More than One Concentrated Load.—An approximate method which may be applied to a beam carrying any number of concentrated loads will now be considered. A particular example is illustrated in Fig. 162, where a beam carries loads W_1 and W_2 and the statical deflections at these loads are δ_1 and δ_2 respectively.

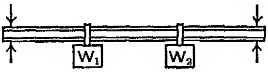

Fig. 162.

It will be assumed that the weight of the beam may be neglected, that the loads vibrate with simple harmonic motion of the same frequency, that the amplitude of the motion of each load is r times its statical deflection, and that each passes through its position of statical equilibrium at the same time.

The method to be employed consists of equating the sum of the kinetic energies of the moving loads at their mean positions to the sum of the strain energies stored in the beam, due to vibration, when each load is at its

maximum deflected position. The rotational movement of each load about an axis perpendicular to the plane of vibration will be neglected.

First consider one load W having a statical deflection δ and therefore, by assumption, an amplitude $r\delta$. Since deflection is proportional to load, it follows that a deflection $r\delta$ could be produced by a force rW. In Fig. 163 the load W is represented diagrammatically in its mean position by a black dot and the amplitude $r\delta$ is shown greatly magnified. As indicated in the Fig., if a radius $OA = r\delta$ rotates with an angular velocity ω rad./sec., then the projection P of A on the diameter YOY′ will have a maximum velocity $v = \omega r\delta$ when it is at the mid-point O, and this is the maximum velocity of the load W.

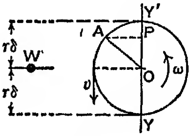

FIG. 163.

The kinetic energy of the load W when in its mid-position is

$$\frac{W}{2g}v^2 = \frac{W}{2g}\omega^2 r^2 \delta^2.$$

The strain energy stored in the beam, *due to the vibration*, when the load is in an extreme position is $\frac{1}{2}rW . r\delta$ or $\frac{1}{2}Wr^2\delta$.

Equating these energies,

$$\frac{W}{2g}\omega^2 r^2 \delta^2 = \frac{1}{2}Wr^2\delta,$$

from which $$\omega = \sqrt{\frac{gW\delta}{W\delta^2}} \text{ rad./sec.,}$$

and for one load this leads to the results obtained in the preceding Art.

For a number of loads the frequency becomes

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g\Sigma W\delta}{\Sigma W\delta^2}} \text{ oscillations/sec.}$$

For the particular example shown in Fig. 162,

$$f = \frac{1}{2\pi}\sqrt{\frac{g(W_1\delta_1 + W_2\delta_2)}{W_1\delta_1^2 + W_2\delta_2^2}} \text{ oscillations/sec.}$$

In applying this method to a numerical example, the major part of the work often depends on finding the values of the deflections under the loads. Another way of solving this problem is by Dunkerley's formula given in Art. 94 on the Whirling of Shafts.

92. Transverse Vibrations of an Unloaded Beam Simply Supported at Each End.—Consider a uniform beam of length l, weighing w lb. per unit length, simply supported at each end (Fig. 164 (a)). When the beam is vibrating let y be the deflection from the mean position, at time t, of any point on the centre line of the beam at a distance x from one end, as shown at (b) where the deflection is exaggerated. The deflection of the beam due to its weight need not be considered, and it is to be understood that the deflection y is entirely due to vibration. It will be supposed that the vibration is caused by a force, or series of forces, deflecting the beam and then being removed suddenly. Time t will be measured from the instant when the applied forces are removed.

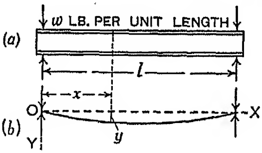

FIG. 164.

The inertia force on a length δx at time t is $\dfrac{w\delta x}{g}\dfrac{\partial^2 y}{\partial t^2}$.

The elastic force on the same length depends on the deflection and is $-\mathrm{EI}\dfrac{\partial^4 y}{\partial x^4}\delta x$ (Art. 97, p. 207), the sign being negative because the elastic force is always directed towards the mean position. The constant E is the modulus of elasticity of the material, and I is the moment of inertia or second moment of area, about the neutral axis, of a cross-section of the beam.

Equating the elastic and inertia forces and dividing by δx,

$$-\mathrm{EI}\frac{\partial^4 y}{\partial x^4}=\frac{w}{g}\frac{\partial^2 y}{\partial t^2},$$

or
$$\frac{\partial^4 y}{\partial x^4}+\frac{w}{g\mathrm{EI}}\frac{\partial^2 y}{\partial t^2}=0 \quad . \quad . \quad . \quad (1).$$

Assume that every point on the beam moves with harmonic motion and let

$$y=\mathrm{X}\cos\omega t \quad . \quad . \quad . \quad (2),$$

where X is a function of x only, ω is in rad./sec., and t is time in sec., then from (2), by partial differentiation,

$$\frac{\partial^4 y}{\partial x^4}=\frac{d^4\mathrm{X}}{dx^4}\cos\omega t \quad \text{and} \quad \frac{\partial^2 y}{\partial t^2}=-\omega^2\mathrm{X}\cos\omega t.$$

Substituting these values in (1),

$$\frac{d^4\mathrm{X}}{dx^4}\cos\omega t-\frac{w}{g\mathrm{EI}}\omega^2\mathrm{X}\cos\omega t=0,$$

then, dividing by $\cos\omega t$ and putting $(w/g\mathrm{EI})\omega^2=a^4$ for convenience in the work which follows,

$$\frac{d^4\mathrm{X}}{dx^4}-a^4\mathrm{X}=0 \quad . \quad . \quad . \quad (3).$$

The general solution of this equation * is

$$\mathrm{X}=\mathrm{A}\cosh ax+\mathrm{B}\sinh ax+\mathrm{C}\cos ax+\mathrm{D}\sin ax \quad (4).$$

Substituting this value of X in (2),

$$y=(\mathrm{A}\cosh ax+\mathrm{B}\sinh ax+\mathrm{C}\cos ax+\mathrm{D}\sin ax)\cos\omega t \quad (5),$$

and the arbitrary constants A, B, C, and D may be found by considering the following conditions, which are true for all values of t: y and $\frac{\partial^2 y}{\partial x^2}$ are zero at each end of the beam, $\frac{\partial^2 y}{\partial x^2}$ being proportional to the bending moment.

Differentiating (5) twice with respect to x,

$$\frac{\partial^2 y}{\partial x^2}=a^2(\mathrm{A}\cosh ax+\mathrm{B}\sinh ax-\mathrm{C}\cos ax-\mathrm{D}\sin ax)\cos\omega t \quad (6).$$

* See the author's *Mathematics* or any book on Differential Equations.

Putting $x=0$ and $y=0$ in (5),

$$0=\mathrm{A}+\mathrm{C} \quad . \quad . \quad . \quad (7).$$

Putting $x=0$ and $\dfrac{\partial^2 y}{\partial x^2}=0$ in (6),

$$0=\mathrm{A}-\mathrm{C} \quad . \quad . \quad . \quad (8).$$

From (7) and (8), by addition and subtraction, it follows that $\mathrm{A}=0$ and $\mathrm{C}=0$.

Putting $x=l$, $y=0$, and $\mathrm{A}=\mathrm{C}=0$ in (5),

$$0=\mathrm{B}\sinh al+\mathrm{D}\sin al \quad . \quad . \quad (9).$$

Putting $x=l$, $\dfrac{\partial^2 y}{\partial x^2}=0$, and $\mathrm{A}=\mathrm{C}=0$ in (6),

$$0=\mathrm{B}\sinh al-\mathrm{D}\sin al \quad . \quad . \quad (10).$$

From (9) and (10), by addition,

$$2\mathrm{B}\sinh al=0.$$

Either $\mathrm{B}=0$ or $\sinh al=0$. If $\sinh al=0$ then $al=0$. But al cannot be zero when the beam is vibrating because $(w/g\mathrm{EI})\omega^2=a^4$ and ω is not zero; therefore $\mathrm{B}=0$.

Equation (5) now reduces to

$$y=\mathrm{D}\sin ax\cos\omega t \quad . \quad . \quad (11).$$

From (9) and (10), by subtraction,

$$2\mathrm{D}\sin al=0.$$

Either $\mathrm{D}=0$ or $\sin al=0$. If $\mathrm{D}=0$, then y is always zero in (11) and this means that the beam is not vibrating. Therefore $\sin al=0$.

Therefore $\qquad al=\pi,\ 2\pi,\ 3\pi$, etc.,

or $\qquad a=\pi/l,\ 2\pi/l,\ 3\pi/l$, etc.

But $\qquad a^4=\dfrac{w}{g\mathrm{EI}}\omega^2$ from which $\omega=a^2\sqrt{\dfrac{g\mathrm{EI}}{w}}$.

Therefore ω may have the following series of values:

$$\omega_1=\frac{\pi^2}{l^2}\sqrt{\frac{g\mathrm{EI}}{w}}, \qquad \omega_2=4\frac{\pi^2}{l^2}\sqrt{\frac{g\mathrm{EI}}{w}}=4\omega_1,$$

$$\omega_3=9\frac{\pi^2}{l^2}\sqrt{\frac{g\mathrm{EI}}{w}}=9\omega_1, \quad \text{and so on.}$$

Substituting the lowest values of a and ω in (11),

$$y = \mathrm{D} \sin \frac{\pi x}{l} \cos \frac{\pi^2}{l^2}\sqrt{\frac{g\mathrm{EI}}{w}}t \quad . \quad . \quad (12),$$

and this gives the fundamental or first mode of vibration. The first three modes of vibration are shown in Fig. 165.

The fundamental periodic time is

$$\mathrm{T} = \frac{2\pi}{\omega_1} = \frac{2l^2}{\pi}\sqrt{\frac{w}{g\mathrm{EI}}} \text{ sec.}$$

The fundamental frequency is

$$f = \frac{1}{\mathrm{T}} = \frac{\pi}{2l^2}\sqrt{\frac{g\mathrm{EI}}{w}} \text{ oscillations/sec.}$$

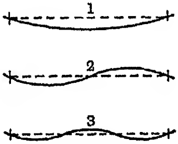

Fig. 165.

Example.—Given $l = 10$ ft., $w = 9$ lb. per ft., $\mathrm{I} = 10{\cdot}91$ in.^{4}, and $\mathrm{E} = 30 \times 10^6$ lb./in.^{2}, to find the fundamental frequency.

Working in lb., ft., and sec. units,

$$f = \frac{\pi}{2 \times 10^2}\sqrt{\frac{32{\cdot}2 \times 30 \times 10^6 \times 12^2 \times 10{\cdot}91}{9 \times 12^4}}$$

$$= 44{\cdot}8 \text{ oscillations/sec.}$$

93. Whirling of Shafts—Single Loads.—When a shaft rotates there are various speeds at which it deflects, and these deflections become dangerous unless the speed of rotation is quickly altered. These particular speeds are called *critical speeds* or *whirling speeds.* When a shaft *whirls*, the centrifugal forces (see Art. 43, p. 73) just exceed the elastic righting forces in the shaft itself, and the whirling speeds may be found by equating these forces. The simplest case occurs when a shaft carries a flywheel which is heavy compared with the weight of the shaft, then, neglecting the latter weight, the shaft will have one whirling speed.

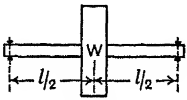

Fig. 166.

A particular case is shown in Fig. 166, where a flywheel of weight W is secured to the centre of a shaft, of length l, which is supported at each end in a short bearing. A short bearing

will be interpreted as one which allows the shaft to deflect as if freely supported.

Let ω be the whirling speed, let y be the deflection of the flywheel from the position of statical equilibrium, and let k be the force which would cause unit deflection.

Now deflection is proportional to load, and a deflection y would be caused by a force ky and this is the elastic force in the shaft. The centrifugal force is $\frac{W}{g}\omega^2 y$.

Equating these forces,

$$\frac{W}{g}\omega^2 y = ky,$$

from which $$\omega = \sqrt{\frac{kg}{W}}.$$

This result may be written as

$$\omega = \sqrt{\frac{g}{\delta}},$$

where $\delta = W/k$ is the static deflection due to the load W. If g is in in./sec.^{2} and δ is in inches, then ω will be in rad./sec.

The formula may also be expressed as

$$\text{Whirling speed} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} \text{ rev./sec.},$$

and it will be noticed that this is the same form as that obtained in Art. 90 for the transverse frequency of a beam carrying one load.

Since shaft speeds are generally reckoned in revolutions per minute, it is more convenient to give the whirling speed as

$$N = \frac{30}{\pi}\sqrt{\frac{g}{\delta}} \text{ rev./min.}$$

The formula may be applied to any shaft carrying one flywheel which is heavy compared with the weight of the shaft.

For the case shown in Fig. 166, the static deflection of the shaft at the central load W is $\delta = Wl^3/48EI$, where E is the modulus of elasticity and I is the moment of inertia, about a diameter, of a cross-section of the shaft.

Example.—Suppose $W = 500$ lb., $l = 5$ ft., $E = 30 \times 10^6$ lb./in.^{2}, and the diameter of the shaft is $d = 2$ in., then $I = \pi d^4/64 = \pi/4$ in.^{4},

$$\delta = \frac{Wl^3}{48EI} = \frac{500 \times 5^3 \times 12^3 \times 4}{48 \times 30 \times 10^6 \times \pi} = 0{\cdot}0955 \text{ in.,}$$

and $$N = \frac{30}{\pi}\sqrt{\frac{g}{\delta}} = \frac{30}{\pi}\sqrt{\frac{32{\cdot}2 \times 12}{0{\cdot}0955}} = 607 \text{ rev./min.}$$

94. Whirling of Shafts—More than One Load.—It has been shown in the preceding Art. that the whirling speed of a shaft carrying one load has the same numerical value as the natural frequency of transverse vibrations. This also applies to shafts carrying more than one load and to unloaded shafts. Therefore, referring to Art. 91, the first whirling speed of a shaft carrying several loads may be given, approximately, as

$$\text{Whirling speed} = \frac{1}{2\pi}\sqrt{\frac{g\Sigma W\delta}{\Sigma W\delta^2}} \text{ rev./sec.,}$$

or $$N = \frac{30}{\pi}\sqrt{\frac{g\Sigma W\delta}{\Sigma W\delta^2}} \text{ rev./min.} \quad . \quad . \quad (1).$$

There is also an empirical formula due to Dunkerley which may be used. Suppose a shaft carries loads W_1, W_2, W_3, etc., and let N_1, N_2, N_3, etc., be the corresponding whirling speeds when the loads are considered one at a time. Also let N be the whirling speed when the shaft carries all the loads, then, according to Dunkerley's formula,

$$\frac{1}{N^2} = \frac{1}{N_1^2} + \frac{1}{N_2^2} + \frac{1}{N_3^2} + \ldots \quad . \quad . \quad (2).$$

The two methods are illustrated by the example which follows.

Example.—To find the whirling speed in revolutions per minute of a steel shaft which is loaded and supported in short bearings as shown in Fig. 167, where the linear dimensions are in inches. The weight of the shaft is to be neglected.

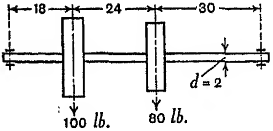

FIG. 167.

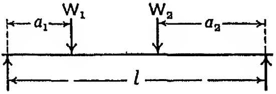

FIG. 168.

First Method.—The static deflection of the shaft is required at each load when both loads are acting. In many complicated examples these deflections have to be found graphically, but in the present case formulæ will be used.

Referring to Fig. 168 and denoting the deflections under the loads W_1 and W_2 by δ_1 and δ_2 respectively, it can be shown for a shaft with freely supported ends that

$$\delta_1 = \frac{a_1}{6EIl}\{2a_1W_1(l-a_1)^2 + a_2W_2(l^2 - a_1^2 - a_2^2)\},$$

(see Ex. 12, p. 239) and then, from symmetry, interchanging the suffixes 1 and 2,

$$\delta_2 = \frac{a_2}{6EIl}\{2a_2W_2(l-a_2)^2 + a_1W_1(l^2 - a_1^2 - a_2^2)\}.$$

The value of

$$I = \pi d^4/64 = \pi \times 2^4/64 = \pi/4 \text{ in.}^4 \quad \text{and} \quad E = 30 \times 10^6 \text{ lb./in.}^2.$$

Substituting numerical values, working in lb. and inch units,

$$l^2 - a_1^2 - a_2^2 = 72^2 - 18^2 - 30^2 = 3960,$$

$$\delta_1 = \frac{18 \times 4}{6 \times 30 \times 10^6 \times \pi \times 72}\{2 \times 18 \times 100 \times 54^2 + 30 \times 80 \times 3960\}$$

$$= 0{\cdot}0354 \text{ in.}$$

and

$$\delta_2=\frac{30\times 4}{6\times 30\times 10^6\times\pi\times 72}\{2\times 30\times 80\times 42^2+18\times 100\times 3960\}$$

$$=0{\cdot}0460 \text{ in.}$$

Whirling speed

$$N=\frac{30}{\pi}\sqrt{\frac{g\Sigma W\delta}{\Sigma W\delta^2}}=\frac{30}{\pi}\sqrt{\frac{g(W_1\delta_1+W_2\delta_2)}{W_1\delta_1^2+W_2\delta_2^2}} \text{ rev./min.}$$

$$=\frac{30}{\pi}\sqrt{\frac{32{\cdot}2\times 12(100\times 0{\cdot}0354+80\times 0{\cdot}0460)}{100\times 0{\cdot}0354^2+80\times 0{\cdot}0460^2}}$$

$$=929 \text{ rev./min.}$$

Second Method.—When using this method the loads are considered one at a time and a formula is required for the deflection, under the load, of a shaft carrying one load.

In Fig. 169 (A) a load W is carried on a shaft of length $a+b$ at a distance a from one end and the shaft is freely supported at each end. Denoting the deflection under the load by δ, it can be shown (Art. 106, p. 221) that

$$\delta=\frac{Wa^2b^2}{3(a+b)EI},$$

and, as before, $E=30\times 10^6$ lb./in.^{2} and $I=\pi/4$ in.^{4}.

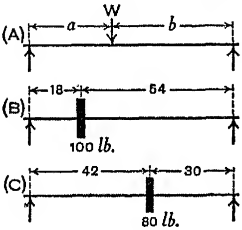

FIG. 169.

Considering the 100-lb. load (Fig. 169 (B)) and denoting the deflection under it by δ_1, then substituting the numerical values in the formula,

$$\delta_1=\frac{Wa^2b^2}{3(a+b)EI}=\frac{100\times 18^2\times 54^2\times 4}{3\times 72\times 30\times 10^6\times\pi}=0{\cdot}0186 \text{ in.}$$

If N_1 is the whirling speed when the shaft carries the 100-lb. load, then

$$N_1=\frac{30}{\pi}\sqrt{\frac{g}{\delta_1}}=\frac{30}{\pi}\sqrt{\frac{32{\cdot}2\times 12}{0{\cdot}0186}}=1380 \text{ rev./min.}$$

Considering the 80-lb. load (Fig. 169 (C)) and denoting the deflection under it by δ_2, then substituting numerical values in the formula,

$$\delta_2 = \frac{Wa^2b^2}{3(a+b)EI} = \frac{80 \times 42^2 \times 30^2 \times 4}{3 \times 72 \times 30 \times 10^6 \times \pi} = 0{\cdot}0250 \text{ in.}$$

If N_2 is the whirling speed when the shaft carries the 80-lb. load, then

$$N_2 = \frac{30}{\pi}\sqrt{\frac{g}{\delta_2}} = \frac{30}{\pi}\sqrt{\frac{32{\cdot}2 \times 12}{0{\cdot}0250}} = 1190 \text{ rev./min.}$$

The whirling speed N when the shaft carries both loads is given by

$$\frac{1}{N^2} = \frac{1}{N_1^2} + \frac{1}{N_2^2} = \frac{1}{1380^2} + \frac{1}{1190^2},$$

from which

$$N = \sqrt{\frac{1380^2 \times 1190^2}{1380^2 + 1190^2}} = 901 \text{ rev./min.}$$

and this result is about 3 per cent. lower than that obtained by the first method.

95. Whirling Speeds of an Unloaded Shaft with a Short Bearing at Each End.—Theoretically, an unloaded shaft will whirl at an infinite number of speeds, but in practice it is not possible to run the shaft fast enough to reach more than a few of these critical speeds, and in many cases it is only the first whirling speed which need be considered.

In Fig. 170 an unloaded shaft, of length l and of uniform diameter d, is shown supported at each end in a short bearing. As previously explained, it will be assumed that a short bearing allows a shaft to deflect as if it were simply supported.

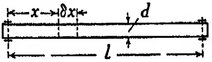

FIG. 170.

Let y be the deflection at a distance x from one end,
E the modulus of elasticity of the material of the shaft,
I the moment of inertia of a cross-section about its neutral axis,
w the weight of the shaft per unit length,
ω the whirling speed,
and g the acceleration due to gravity.

The whirling speed may be found by equating the elastic force and the centrifugal force acting on a length δx. The elastic force is $\mathrm{EI}\frac{d^4y}{dx^4}\delta x$ (Art. 97, p. 207) and the centrifugal force is $\frac{w\delta x}{g}\omega^2 y$, therefore

$$\mathrm{EI}\frac{d^4y}{dx^4}=\frac{w}{g}\omega^2 y,$$

or

$$\frac{d^4y}{dx^4}-\frac{w\omega^2}{g\mathrm{EI}}y=0 \quad . \quad . \quad . \quad (1).$$

Denoting $\frac{w\omega^2}{g\mathrm{EI}}$ by a^4, for convenience in the work which follows, then

$$\frac{d^4y}{dx^4}-a^4y=0 \quad . \quad . \quad . \quad (2),$$

which is the same form as equation (3), Art. 92, obtained when considering the transverse vibrations of a beam, and the solution is

$$y=\mathrm{A}\cosh ax+\mathrm{B}\sinh ax+\mathrm{C}\cos ax+\mathrm{D}\sin ax \qquad (3).$$

The arbitrary constants A, B, C, and D are found as in Art. 92; y and $\frac{d^2y}{dx^2}$ are zero at each end of the shaft, $\frac{d^2y}{dx^2}$ being proportional to the bending moment.

Differentiating (3) twice,

$$\frac{d^2y}{dx^2}=a^2(\mathrm{A}\cosh ax+\mathrm{B}\sinh ax-\mathrm{C}\cos ax-\mathrm{D}\sin ax) \quad (4).$$

Putting $x=0$ and $y=0$ in (3),

$$0=A+C \quad . \quad . \quad . \quad (5).$$

Putting $x=0$ and $\frac{d^2y}{dx^2}=0$ in (4),

$$0=A-C \quad . \quad . \quad . \quad (6).$$

From (5) and (6), by addition and subtraction, it follows that $A=0$ and $C=0$.

Putting $x=l$, $y=0$, and $A=C=0$ in (3),

$$0=B \sinh al+D \sin al \quad . \quad . \quad (7).$$

Putting $x=l$, $\frac{d^2y}{dx^2}=0$, and $A=C=0$ in (4),

$$0=B \sinh al-D \sin al \quad . \quad . \quad (8).$$

From (7) and (8), by addition,

$$2B \sinh al=0.$$

Either $B=0$ or $\sinh al=0$. If $\sinh al=0$, then $al=0$, but l cannot be zero and a cannot be zero when the shaft is whirling, therefore $B=0$.

Equation (3) now reduces to

$$y=D \sin ax \quad . \quad . \quad . \quad (9).$$

From (7) and (8), by subtraction,

$$2D \sin al=0.$$

Either $D=0$ or $\sin al=0$. If $D=0$, then y is zero in (9) and this is impossible for the shaft is whirling, therefore $\sin al=0$.

Therefore $\qquad al=\pi,\ 2\pi,\ 3\pi$, etc.,

and $\qquad a=\pi/l,\ 2\pi/l,\ 3\pi/l$, etc.

But $\qquad a^4=\frac{w\omega^2}{gEI}$, from which $\omega=a^2\sqrt{\frac{gEI}{w}}$.

Therefore ω may have the following values:—

$$\omega_1=\frac{\pi^2}{l^2}\sqrt{\frac{gEI}{w}}, \qquad \omega_2=4\frac{\pi^2}{l^2}\sqrt{\frac{gEI}{w}}=4\omega_1,$$

$$\omega_3=9\frac{\pi^2}{l^2}\sqrt{\frac{gEI}{w}}=9\omega_1,$$

and so on, the units being rad./sec.

Expressing these whirling speeds in rev./min., then

$$N_1 = \frac{60}{2\pi}\omega_1 = 30\frac{\pi}{l^2}\sqrt{\frac{gEI}{w}}, \qquad N_2 = 4N_1, \qquad N_3 = 9N_1,$$

and so on.

Substituting values of a in (9), then

$$y = D \sin\frac{\pi x}{l}, \qquad y = D \sin\frac{2\pi x}{l}, \qquad y = D \sin\frac{3\pi x}{l},$$

and so on. From these equations it is evident that when the shaft whirls it will bend as shown in Fig. 165, p. 191, where 1, 2, and 3 now indicate the first, second, and third whirling speeds respectively.

Example 1.—To find the first and second whirling speeds of a steel shaft 6 feet long, $\frac{3}{4}$ inch diameter, supported in a short bearing at each end. Assume that a cubic inch of steel weighs 0·28 lb. and take $E = 30 \times 10^6$ lb./in.².

The weight of an inch length of shaft of diameter d inches is

$$w = 0{\cdot}28\frac{\pi}{4}d^2 = 0{\cdot}07\pi d^2 \text{ lb.}$$

Also $g = 32{\cdot}2 \times 12$ in./sec.² and $I = \pi d^4/64$.

Substituting numerical values in the formula

$$N_1 = 30\frac{\pi}{l^2}\sqrt{\frac{gEI}{w}},$$

then

$$N_1 = 30\frac{\pi}{l^2}\sqrt{\frac{32{\cdot}2 \times 12 \times 30 \times 10^6}{0{\cdot}07\pi d^2} \times \frac{\pi d^4}{64}} = 4{\cdot}8 \times 10^6\frac{d}{l^2} \text{ rev./min.}$$

where d and l are in inches.

Putting $d = \frac{3}{4}$ and $l = 72$, then

$$N_1 = 4{\cdot}8 \times 10^6 \times \tfrac{3}{4} \times \frac{1}{72^2} = 694 \text{ rev./min.}$$

The second whirling speed is

$$N_2 = 4N_1 = 4 \times 694 = 2776 \text{ rev./min.}$$

Since the value of E was given to two figures only, these results are probably correct to two figures only.

Example 2.—To find the first whirling speed of the shaft described in Ex. 1 if a mass weighing 5 lb. is attached to it midway between the bearings.

First suppose the shaft to be weightless, then the whirling speed due to the 5-lb. load is given by the formula

$$N=\frac{30}{\pi}\sqrt{\frac{g}{\delta}}\ \text{rev./min.} \quad \text{(Art. 93)},$$

where, in this case,

$$\delta=\frac{Wl^3}{48EI}=\frac{Wl^3}{48E}\times\frac{64}{\pi d^4}. \quad \text{(Art. 100.)}$$

Substituting numerical values, using lb. and inch units,

$$\delta=\frac{5\times72^3}{48\times30\times10^6}\times\frac{64\times4^4}{\pi\times3^4}=0{\cdot}0834\ \text{in.},$$

and $$N=\frac{30}{\pi}\sqrt{\frac{32{\cdot}2\times12}{0{\cdot}0834}}=650\ \text{rev./min.}$$

The first whirling speed of the shaft due to its own weight when there is no other load was found in Ex. 1 to be $N_1=694$ rev./min.

The two results may be combined now by Dunkerley's empirical formula given in Art. 94. Denoting the whirling speed by N_c when the 5-lb. load and the weight of the shaft are both taken into account, then

$$\frac{1}{N_c^2}=\frac{1}{N^2}+\frac{1}{N_1^2}=\frac{1}{650^2}+\frac{1}{694^2},$$

from which $$N_c=474\ \text{rev./min.}$$

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Exercises X

When required, assume, unless otherwise stated, that for steel $E = 30 \times 10^6$ *lb./in.*^{2} *and* $C = 11{\cdot}7 \times 10^6$ *lb./in.*^{2}, *and take* $g = 32{\cdot}2$ *ft./sec.*^{2}.

1. A light helical spring, which stretches 1 inch with a load of 2 pounds, is fixed at its upper end. From the lower end hangs another light helical spring which stretches 1 inch with a load of $1\frac{1}{4}$ pounds. To the lower end of the latter spring is attached a body weighing 5 pounds.

Find the period of an oscillation of the body along the axis of the springs. Prove the formula you use. [B.E.]

2. It is found that a given ma[illegible] a fixed point by a light helical spring, falls a [illegible] before rising if it is carefully attached, without stretching the spring, and is then allowed to fall under gravity. Form the equation of motion and deduce the number of vibrations per minute.

The mass when suspended from a second light spring makes 300 vibrations per minute. If the two springs are now joined end on and the mass again suspended, how many vibrations per minute will be made?

If, when the mass is at rest, an equal mass be dropped on to it from a height of one foot, without rebound, find how far the two masses will descend before coming momentarily to rest. [B.E.]

3. A body weighing 5 lb. vibrates with a frequency of 4·9 oscillations per second in a vacuum and a frequency of 4·8 oscillations per second when the motion is damped by a resistance which is proportional to the velocity. Find the resistance per unit velocity.

4. A weight of 20 lb. is attached to a spring requiring 10 lb. to extend it by 1 in. The weight is deflected 6 in. from its equilibrium position and is then allowed to vibrate under the action of a periodic force $60 \cos \frac{\pi}{5}t$ lb., where t is measured from the instant of release.

Draw a curve showing its position at each instant during the first 3 seconds, (*a*) if vibrating in air, (*b*) if vibrating in a very viscous fluid which offers a resistance to motion equal to 10 lb. when the velocity is 1 ft. per second. [U.L.]

5. A body of mass 80·5 lb. has a simple harmonic motion under a force of 31·25 lb. per foot displacement. Find the period.

If the body be retarded by a frictional force proportional to the speed, and equal to 3·75 lb. per ft./sec., find the new period and the log. dec. of the amplitude.

If, further, there is in action a force R sin pt, of period 2 sec., find R in order that the amplitude of the forced oscillation may be 0·5 ft. [B.E.]

6. Show that the inclination θ to the vertical, at any time t, of a simple pendulum of length l making small oscillations in a medium in which the resistance per unit mass is k times the linear velocity is given by

$$l\frac{d^2\theta}{dt^2}+kl\frac{d\theta}{dt}+g\theta=0.$$

The time of a complete oscillation o[illegible] small oscillations *in vacuo* is 2 seconds; if [illegible] due to the air is 0·04 times the angular velocity of the pendulum, and the initial amplitude is 1°, show that in ten complete oscillations the amplitude will be reduced to approximately 40′. [U.L.]

7. A flywheel weighing 55 lb. is suspended by a steel wire 10 feet long and of 0·144 inch diameter (Fig. 171). It is found by experiment that the flywheel makes 20 complete torsional oscillations in 147 seconds. Find the radius of gyration of the flywheel.

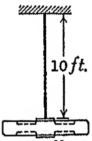

FIG. 171.

8. The crankshaft of an engine carries two disc flywheels 4 feet diameter and each weighing 1 ton. These are mounted 3 feet 6 inches apart on the shaft, which is 4 inches diameter. Determine the time of natural torsional oscillation of the system. Take the modulus of rigidity of the material as $12{\cdot}5\times10^6$ lb. per sq. inch. [U.L.]

9. The shaft shown in Fig. 172 is fixed at one end and carries at the other end a flywheel which weighs 50 lb. and has a radius of gyration of $7\frac{1}{2}$ inches. The diameters and lengths indicated are in inches. Find the natural frequency of torsional oscillation of the system.

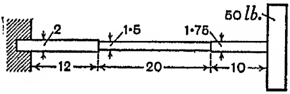

FIG. 172.

10. A shaft, of length l and circular cross-section, tapers uniformly from a diameter d_1 to a diameter d_2. If it is fixed at one end and has a heavy flywheel of moment of inertia I attached to it at the other end, find the value of the periodic time of torsional oscillation.

11. Calculate the lowest frequency of torsional vibration of a steel shaft 20 feet long which is held at one end. Assume the weight of a cubic foot of steel to be 490 lb.

12. A uniform circular plate of 18 inches diameter, mass 12 lb., is suspended horizontally by a vertical wire attached to its centre and fixed at the upper end. The period of a complete torsional oscillation is 2·2 seconds.

Find the couple required to turn the plate through one revolution.

A small body of mass m lb. rests in a groove on the plate. Assuming that the effect of the groove and the added mass on the inertia of the plate is negligible, find the greatest frictional force brought into play to prevent the body sliding when the amplitude of the oscillation is 90°, (i) when the groove is radial and m is 6 inches from the centre, (ii) when the groove is circular, concentric with the plate, and of 6 inches radius. [B.E.]

13. A simply supported girder has a span of 10 feet and its dimensions are such that a single load of 1 ton at mid-span produces a deflection of 1/10th inch. A single-cylinder engine, weighing 300 lb., is mounted on the girder at the centre of the span and is out of balance by an amount equivalent to a weight of 1 lb. at a radius of 4 inches. The frictional and other resistances to vertical movement of the system are proportional to velocity and have a value of 10 lb. for a velocity of 1 foot per second. Find (*a*) the speed of the engine at which maximum deflection occurs in the girder, and (*b*) the maximum mid-span deflection when the engine runs at 200 r.p.m. [U.L.]

14. A disc weighing 9 lb. is mounted with its axis horizontal at the middle of a light beam which deflects 3·85 inches under the weight. Calculate the natural period of vibration of the system.

If the disc is lifted, so that the weight is just taken from the beam, and then released, draw a curve showing the position of the weight during the first two seconds of movement (1) when there is no damping, (2) when the motion is damped by a resistance proportional to the velocity, having a value of 4·2 lb. at 1 ft. per sec.

If the C.G. of the disc is $1\frac{1}{8}$ inches out of truth, and the disc is rotated at speeds increasing from zero to 200 r.p.m., the motion being damped, draw a curve showing the maximum amplitude of vibration at each speed. Draw a curve showing how the difference of phase between the out-of-balance force and the displacement of the disc varies with the speed. [U.L.]

15. An unloaded beam is fixed at one end and is quite free at the other end. Using the notation of Art. 92, show that $\cosh al = -\sec al$ when transverse vibrations occur; then,

solving this equation graphically, show that the lowest value of al is 1·875. [*Hint.*—When $x=0$, then $y=0$ and $\frac{\partial y}{\partial x}=0$; when $x=l$, then $\frac{\partial^2 y}{\partial x^2}=0$ and $\frac{\partial^3 y}{\partial x^3}=0$.]

Given $l=10$ ft., $w=9$ lb. per ft., $I=10{\cdot}91$ in.^{4}, and $E=30\times10^6$ lb./in.^{2}, find the fundamental frequency of vibration. Compare this frequency with that obtained in the example worked out on p. 191.

16. A steel shaft 8 feet long and 3 inches diameter is supported at each end in a short bearing and carries a disc weighing $\frac{1}{2}$ ton at a distance of 3 feet from one end. Find the whirling speed, neglecting the weight of the shaft. (The deflection formula required is $\delta=\frac{Wa^2b^2}{3(a+b)EI}$.)

17. A steel shaft is loaded and supported in short bearings as shown in Fig. 173, where the linear dimensions are in inches. Neglecting the weight of the shaft, find the lowest whirling speed by each of the methods described in Art. 94.

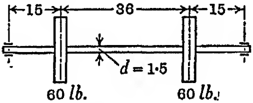

FIG. 173.

18. Obtain an expression for the first whirling speed of a uniform shaft supported freely at its ends and unloaded; also an expression for the whirling speed of a weightless uniform shaft with a central load; then combine the two expressions.

Find the first whirling speed of a shaft, 0·5 inch in diameter and 24 inches in length, carrying a load of 2·25 pounds at its middle point. The metal of the shaft weighs 0·3 lb./in.^{3} and $E=30\times10^6$ lb./in.^{2}. [U.L.]

19. If the ends of a shaft are constrained in long bearings so that the slope at each end is zero, show, using the notation of Art. 95, that the shaft will whirl when $\cosh al=\sec al$. By solving this equation graphically, show that at the lowest whirling speed $al=4{\cdot}73$ approximately.

20. In order to find the moment of inertia of an accurately balanced turbine disc of mass M lb. keyed to a shaft of radius r ft., the disc is placed with the shaft horizontal and resting on level surfaces on either side, and a small mass m lb. is fixed to the wheel at a distance h ft. from the axis of the shaft. If T seconds is the time of a complete small oscillation, prove that the moment of inertia of the wheel is given by

$$I=m\left\{\frac{ghT^2}{4\pi^2}-(h-r)^2\right\}-Mr^2. \qquad \text{[U.L.]}$$

21. Prove that the period of oscillation of the sleeve of a Porter governor in which the speed of revolution is ω rad./sec. is

$$T = \frac{2\pi}{\omega}\left\{\frac{m + 2M\sin^2\alpha}{m} \cdot \frac{l\sin\alpha}{l\sin^3\alpha + c}\right\}^{\frac{1}{2}}$$

if the sleeve is slightly disturbed from its position of equilibrium corresponding to this speed. The four links are each equal in length to l, and are pivoted at points distance c from the axis of rotation, and are, for the configuration corresponding to the speed ω, inclined at α to the vertical. The mass of each ball is m and of the central load M. [U.L.]

CHAPTER XI

DEFLECTION OF BEAMS

96. **Radius of Curvature.**—Suppose the curve in Fig. 174 is the graph of an equation in which y is given as some function of x—that is, $y=f(x)$. Let P and Q be points near together on the curve, the co-ordinates of P being x and y, and the co-ordinates of Q being $x+\delta x$ and $y+\delta y$.

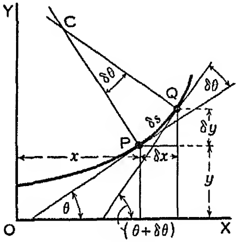

FIG. 174.

Draw the tangents at P and Q and let these tangents make angles θ and $\theta+\delta\theta$, respectively, with the axis OX and consequently an angle $\delta\theta$ with one another. Draw the normals PC and QC, intersecting at C, then the angle PCQ is equal to $\delta\theta$.

Let the length PQ, measured along the curve, be δs, then

$$CP\,.\,\delta\theta=\delta s \quad \text{or} \quad \frac{1}{CP}=\frac{\delta\theta}{\delta s} \quad \text{approximately.}$$

When Q is indefinitely near to P, then CP becomes the *radius of curvature* of the curve at the point P and will be denoted by the symbol R, then

$$\frac{1}{R}=\frac{d\theta}{ds},$$

which may also be written as

$$\frac{1}{R}=\frac{d\theta}{dx}\,\frac{dx}{ds} \qquad . \quad . \quad . \quad (1).$$

At the point P, $\frac{dy}{dx}=\tan\theta$, therefore, differentiating with respect to x,

$$\frac{d^2y}{dx^2}=\sec^2\theta\,\frac{d\theta}{dx}=(1+\tan^2\theta)\frac{d\theta}{dx}=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{d\theta}{dx},$$

from which $$\frac{d\theta}{dx}=\frac{d^2y}{dx^2}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{-1} \quad . \quad . \quad (2).$$

Since $(\delta s)^2=(\delta x)^2+(\delta y)^2$ approximately, therefore, dividing by $(\delta x)^2$ and taking the square root of each side of the equation,

$$\frac{\delta s}{\delta x}=\left\{1+\left(\frac{\delta y}{\delta x}\right)^2\right\}^{\frac{1}{2}} \text{ approximately,}$$

and in the limit $$\frac{ds}{dx}=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$$

or $$\frac{dx}{ds}=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{-\frac{1}{2}} \quad . \quad . \quad . \quad (3).$$

From (1), (2), and (3),

$$\frac{1}{R}=\frac{d^2y}{dx^2}\bigg/\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} \quad . \quad . \quad (4).$$

When $\frac{dy}{dx}$ is very small, as in the case of a slightly deflected beam which was straight initially, then $\left(\frac{dy}{dx}\right)^2$ is negligible compared with unity, and equation (4) may be written

$$\frac{1}{R}=\frac{d^2y}{dx^2} \text{ approximately} \quad . \quad (5).$$

97. Deflection of Beams.—In Fig. 175 AB is a beam deflected downwards from its initial position along AX by some applied load or loads, the amount of deflection being exaggerated. Taking the origin O at A, or wherever convenient, then at any point P on the beam the distance from the axis OY is x and the distance below the

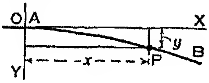

FIG. 175.

axis OX is y. This distance y is known as the deflection at the point P. The sign of y will be taken as positive when the deflection is downwards.

At any cross-section of a beam, it is known that $\frac{E}{R}=\frac{M}{I}$ (see *Applied Mechanics*, Arts. 109 and 110, pp. 103–105),

where E is the modulus of elasticity,
R is the radius of curvature of the neutral surface,
M is the bending moment,
and I is the moment of inertia, about the neutral axis, of the cross-section.

But, as shown in the preceding Art., $\frac{1}{R}=\frac{d^2y}{dx^2}$, therefore

$$\frac{d^2y}{dx^2}=\frac{M}{EI} \quad . \quad . \quad . \quad . \quad (1).$$

The question of signs must now be considered. In Figs. 174 and 175 the concave side of the curve is towards the positive direction of y, and it is evident that a hogging bending moment produces this effect in Fig. 175. Therefore, if a hogging bending moment is regarded as positive, equation (1) is correct. However, in the *Applied Mechanics*, p. 88, a sagging bending moment is taken as positive and, following this convention, equation (1) must be written

$$\frac{d^2y}{dx^2}=-\frac{M}{EI} \quad . \quad . \quad . \quad (2).$$

The procedure in finding the deflection at any section of a beam is quite straightforward. The bending moment at any section is expressed in terms of x, then $-M/EI$ is integrated twice with respect to x. The first integration gives $\frac{dy}{dx}$, the slope, the second integration gives y, the deflection, and in each case the value at a particular section may be obtained by substituting the corresponding value of x.

Equation (2) may be put into other forms which are

sometimes useful. It can be shown that $\frac{dM}{dx}=F$ and $\frac{dF}{dx}=-w$, where F is shearing force and w is the distributed load per unit length.

Therefore differentiating (2), taking I to be a constant,

$$\frac{d^3y}{dx^3}=-\frac{1}{EI}\frac{dM}{dx}=-\frac{F}{EI} \quad . \quad . \quad (3),$$

then
$$\frac{d^4y}{dx^4}=-\frac{1}{EI}\frac{dF}{dx}=\frac{w}{EI} \quad . \quad . \quad . \quad (4).$$

Starting with the load, using equation (4), the deflection is obtained after integrating four times. Since each integration introduces a constant which has to be determined, it is quicker to use equation (2) whenever possible.

It should be noted for reference that $w=EI\frac{d^4y}{dx^4}$. In the whirling of a shaft (Art. 95, p. 196), the load per unit length is not a static load w but is assumed to be due to centrifugal force and is equated to $EI\frac{d^4y}{dx^4}$, which is regarded as the elastic righting force per unit length in the deflected shaft.

The direction of this elastic force is towards the mean position and its sign must be taken into account when required, as in the transverse vibrations of a beam (Art. 92, p. 188) where the negative sign is used.

A number of deflection examples are worked out in the Arts. which follow. As soon as the student is familiar with the method, he should work out each case himself and compare his solutions with those given in the text. It is to be assumed, unless otherwise stated, that I is a constant for each beam, that the beams are horizontal and weightless, that all supports are at the same level, and that the slope is zero where the beam is built-in or direction fixed. A built-in beam is sometimes referred to as being encastered or *encastré*.

The deflection due to the weight of a beam is equivalent

to the deflection due to a uniformly distributed load carried by a weightless beam, and may be taken into account when necessary.

Lastly, it may be pointed out that the deflection at any section due to several loads on a beam is equal to the algebraic sum of the deflections at the section when the loads are on the beam one at a time.

98. Cantilever Loaded at Free End.—A cantilever of length l carries a load W at the free end (Fig. 176).

Taking the origin O at the fixed end, then at a section s a distance x from O the bending moment is

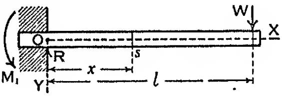

FIG. 176.

$$M = -W(l - x),$$

the sign being negative because the bending moment causes hogging. Since the reaction at the fixed end is $R = W$, therefore, denoting the fixing moment by M_1, the bending moment at the section s may also be written as $M = Wx - M_1$; but $M_1 = Wl$ and so $M = Wx - Wl = -W(l - x)$ as before. This may seem obvious, but it is a common mistake to forget about the fixing moment M_1, and the student must understand that the bending moment is the same whichever side of the section is considered.

Substituting the bending moment in the equation

$$\frac{d^2y}{dx^2} = -\frac{M}{EI},$$

then

$$\frac{d^2y}{dx^2} = \frac{W}{EI}(l - x).$$

Integrating,

$$\frac{dy}{dx} = \frac{W}{EI}(lx - \tfrac{1}{2}x^2 + A),$$

where A is a constant of integration. Assuming that when $x = 0$, the slope $\frac{dy}{dx} = 0$, then substituting these values gives $A = 0$, therefore

$$\frac{dy}{dx} = \frac{W}{EI}(lx - \tfrac{1}{2}x^2)$$

and the slope at any section may be found by substituting the appropriate value of x.

Integrating, $$y=\frac{W}{EI}\left(\tfrac{1}{2}lx^2-\tfrac{1}{6}x^3+B\right),$$

where B is a constant of integration. When $x=0$, the deflection $y=0$, and substituting these values gives $B=0$, therefore

$$y=\frac{W}{EI}\left(\tfrac{1}{2}lx^2-\tfrac{1}{6}x^3\right)$$

and the deflection y at any section may be found by substituting the appropriate value of x.

The maximum deflection occurs when $x=l$, therefore

$$y_{max}=\frac{W}{EI}\left(\tfrac{1}{2}l^3-\tfrac{1}{6}l^3\right)=\frac{Wl^3}{3EI}.$$

99. **Cantilever Loaded Uniformly.**—A cantilever of length l supports a uniformly distributed load w per unit length (Fig. 177).

Taking the origin O at the fixed end, then at a section s a distance x from O the bending moment is

$$M=-\frac{w(l-x)^2}{2}$$

which is obtained by considering the part of the beam to the right of the section s. Considering the part of the beam to the left of the section s, if the fixing moment is M_1 and the reaction is R, then $M_1=\tfrac{1}{2}wl^2$ and $R=wl$, and the bending moment at the section s is $M=Rx-\tfrac{1}{2}wx^2-M_1$, which reduces to the value already obtained.

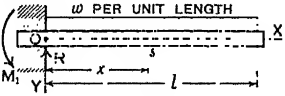

FIG. 177.

Substituting the value of M in the equation $\frac{d^2y}{dx^2}=-\frac{M}{EI}$,

then $$\frac{d^2y}{dx^2}=\frac{w}{2EI}(l^2-2lx+x^2).$$

Integrating, $\dfrac{dy}{dx}=\dfrac{w}{2EI}(l^2x - lx^2+\frac{1}{3}x^3+A)$

where A is a constant of integration. When $x=0$ it is assumed that the slope $\dfrac{dy}{dx}=0$, and substituting these values gives $A=0$, therefore

$$\frac{dy}{dx}=\frac{w}{2EI}(l^2x - lx^2+\tfrac{1}{3}x^3),$$

and from this equation the slope may be found at any section of the beam.

Integrating, $y=\dfrac{w}{2EI}(\frac{1}{2}l^2x^2 - \frac{1}{3}lx^3+\frac{1}{12}x^4+B)$,

where B is a constant of integration. When $x=0$, $y=0$, and substituting these values gives $B=0$, therefore

$$y=\frac{w}{2EI}(\tfrac{1}{2}l^2x^2 - \tfrac{1}{3}lx^3+\tfrac{1}{12}x^4),$$

and from this equation the deflection may be found at any section of the beam.

The deflection is a maximum when $x=l$, therefore

$$y_{max}=\frac{w}{2EI}(\tfrac{1}{2}l^4 - \tfrac{1}{3}l^4+\tfrac{1}{12}l^4)=\frac{wl^4}{8EI}.$$

If the total load wl is denoted by W, then

$$y_{max}=\frac{Wl^3}{8EI}.$$

100. Beam Supported at the Ends and Loaded at the Centre.—A beam simply supported at each end carries a central load W (Fig. 178); the distance between the supports is l.

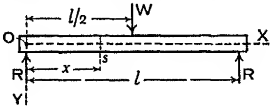

FIG. 178.

Since the load is at the centre, each reaction is $R=\frac{1}{2}W$. Taking the origin O at the left-hand support, then at a section s a distance x from O the bending moment is $M=\frac{1}{2}Wx$, provided x is less than $\frac{1}{2}l$.

Therefore $$\frac{d^2y}{dx^2} = -\frac{M}{EI} = -\frac{W}{2EI}x.$$

Integrating, $$\frac{dy}{dx} = -\frac{W}{2EI}(\tfrac{1}{2}x^2 + A),$$

where A is a constant of integration. When $x = \frac{1}{2}l$, the slope $\frac{dy}{dx} = 0$, therefore $0 = \frac{1}{8}l^2 + A$ and $A = -\frac{1}{8}l^2$, therefore

$$\frac{dy}{dx} = -\frac{W}{4EI}(x^2 - \tfrac{1}{4}l^2).$$

Integrating, $$y = -\frac{W}{4EI}(\tfrac{1}{3}x^3 - \tfrac{1}{4}l^2x + B),$$

where B is a constant of integration. When $x = 0$, $y = 0$, therefore $B = 0$, therefore

$$y = -\frac{W}{4EI}(\tfrac{1}{3}x^3 - \tfrac{1}{4}l^2x).$$

This equation will give the deflection for values of x from 0 to $\frac{1}{2}l$, and this is all that is required since the deflected beam is symmetrical about the centre.

The deflection is a maximum at the centre, therefore putting $x = \frac{1}{2}l$,

$$y_{max} = -\frac{W}{4EI}(\tfrac{1}{24}l^3 - \tfrac{1}{8}l^3) = \frac{Wl^3}{48EI}.$$

101. Beam Supported at the Ends and Loaded Uniformly.—The conditions are the same as in the preceding case, except that the load is w per unit length, making a total load wl. Each reaction is $R = \frac{1}{2}wl$ (Fig. 179).

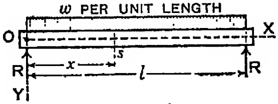

FIG. 179.

The bending moment M at a section s is

$$M = \tfrac{1}{2}wlx - \tfrac{1}{2}wx^2,$$

and this applies for all values of x.

Therefore $$\frac{d^2y}{dx^2}=-\frac{M}{EI}=-\frac{w}{2EI}(lx-x^2).$$

Integrating, $$\frac{dy}{dx}=-\frac{w}{2EI}(\tfrac{1}{2}lx^2-\tfrac{1}{3}x^3+A).$$

When $$x=\tfrac{1}{2}l,\quad \frac{dy}{dx}=0,$$

therefore $$0=\tfrac{1}{8}l^3-\tfrac{1}{24}l^3+A,\quad \text{and}\quad A=-\tfrac{1}{12}l^3,$$

therefore $$\frac{dy}{dx}=-\frac{w}{2EI}(\tfrac{1}{2}lx^2-\tfrac{1}{3}x^3-\tfrac{1}{12}l^3).$$

Integrating, $$y=-\frac{w}{2EI}(\tfrac{1}{6}lx^3-\tfrac{1}{12}x^4-\tfrac{1}{12}l^3x+B).$$

When $x=0$, $y=0$, therefore $B=0$,

therefore $$y=-\frac{w}{12EI}(lx^3-\tfrac{1}{2}x^4-\tfrac{1}{2}l^3x)$$

and this equation applies for all values of x.

The deflection is a maximum at the centre, therefore putting $x=\tfrac{1}{2}l$,

$$y_{max}=-\frac{w}{12EI}(\tfrac{1}{8}l^4-\tfrac{1}{32}l^4-\tfrac{1}{4}l^4)=\frac{5wl^4}{384EI}.$$

If the total load wl is written as W, then $y_{max}=\dfrac{5Wl^3}{384EI}$.

102. Beam Fixed at the Ends and Loaded at the Centre.—A beam AB is fixed at each end and carries a central load W (Fig. 180). The distance between the supports is l. It will be assumed here, and in similar cases, that the ends are fixed in direction only, so that they remain horizontal but are free to move towards one another.

From symmetry it follows that the reactions R are equal and that the fixing moments M_1 are equal. The beam will deflect as shown in the lower part of Fig. 180, the slope being zero at A and B and at the centre C; and at points E and F, equidistant from the centre, the slope

will have its greatest values. The parts AE and BF are concave downwards and the part ECF is concave upwards. The points E and F, where the curvature changes, are called *points of inflexion.* The positions of the points E and F and the deflections at C, E, and F will now be determined.

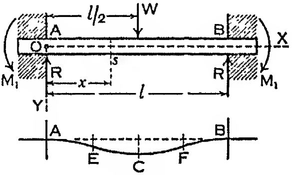

FIG. 180.

Taking the origin at A, then at a section s, a distance x from A, the bending moment is $M = Rx - M_1$. Now $2R = W$ or $R = \frac{1}{2}W$, therefore

$$M = \tfrac{1}{2}Wx - M_1,$$

and this equation holds for values of x from 0 to $\frac{1}{2}l$.

$$\text{Therefore} \quad \frac{d^2y}{dx^2} = -\frac{M}{EI} = -\frac{1}{EI}(\tfrac{1}{2}Wx - M_1).$$

$$\text{Integrating,} \quad \frac{dy}{dx} = -\frac{1}{EI}(\tfrac{1}{4}Wx^2 - M_1x + A')$$

where A' is a constant of integration.

When $x = 0$, $\dfrac{dy}{dx} = 0$, therefore $A' = 0$.

When $x = \frac{1}{2}l$, $\dfrac{dy}{dx} = 0$, therefore $0 = \frac{1}{16}Wl^2 - \frac{1}{2}M_1 l$,

from which $M_1 = \frac{1}{8}Wl$.

$$\text{Therefore} \quad \frac{dy}{dx} = -\frac{W}{4EI}(x^2 - \tfrac{1}{2}lx).$$

The greatest value of $\dfrac{dy}{dx}$ occurs when $\dfrac{d^2y}{dx^2} = 0$, that is when $M = 0$. Therefore $\frac{1}{2}Wx - M_1 = 0$, or $\frac{1}{2}Wx - \frac{1}{8}Wl = 0$, from which $x = \frac{1}{4}l$. This gives the position of the point E, and the point F is at an equal distance from the end B.

Integrating the expression for $\frac{dy}{dx}$ gives

$$y = -\frac{W}{4EI}(\tfrac{1}{3}x^3 - \tfrac{1}{4}lx^2 + B')$$

where B′ is a constant of integration. When $x=0$, $y=0$, therefore $B'=0$ and

$$y = -\frac{W}{4EI}(\tfrac{1}{3}x^3 - \tfrac{1}{4}lx^2),$$

which holds for values of x from 0 to $\frac{1}{2}l$.

The maximum deflection occurs when $x=\frac{1}{2}l$, therefore

$$y_{\text{max}} = -\frac{W}{4EI}(\tfrac{1}{24}l^3 - \tfrac{1}{16}l^3) = \frac{Wl^3}{192EI}.$$

At the point of inflexion E, $x=\frac{1}{4}l$, therefore

$$y_E = -\frac{W}{4EI}(\tfrac{1}{192}l^3 - \tfrac{1}{64}l^3) = \frac{Wl^3}{384EI},$$

and this is also the deflection at the point F.

103. Beam Fixed at the Ends and Loaded Uniformly.—The conditions are as in the preceding case, except that the load is w per unit length, making a total load wl (Fig. 181).

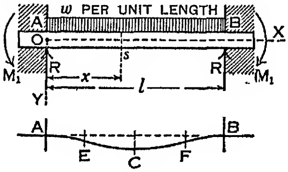

Fig. 181.

Each reaction is $R=\frac{1}{2}wl$. As before, the slope is zero at A, C, and B, and the points of inflexion are at E and F, the lengths AE and BF being equal and unknown.

Taking the origin at the end A, then at a section s, a distance x from A, the bending moment is

$$M = Rx - \tfrac{1}{2}wx^2 - M_1 = \tfrac{1}{2}wlx - \tfrac{1}{2}wx^2 - M_1,$$

and this equation is true for all values of x from 0 to l.

Therefore

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} = -\frac{1}{EI}(\tfrac{1}{2}wlx - \tfrac{1}{2}wx^2 - M_1).$$

Integrating,

$$\frac{dy}{dx} = -\frac{1}{EI}(\tfrac{1}{4}wlx^2 - \tfrac{1}{6}wx^3 - M_1x + A')$$

where A' is a constant of integration.

When $x=0$, $\dfrac{dy}{dx}=0$, therefore $A'=0$.

When $x=l$, $\dfrac{dy}{dx}=0$, therefore $0=\tfrac{1}{4}wl^3 - \tfrac{1}{6}wl^3 - M_1l$

from which $M_1 = \tfrac{1}{12}wl^2$.

Therefore $$\frac{dy}{dx} = -\frac{w}{2EI}(\tfrac{1}{2}lx^2 - \tfrac{1}{3}x^3 - \tfrac{1}{6}l^2x).$$

The greatest value of $\dfrac{dy}{dx}$ occurs when $\dfrac{d^2y}{dx^2}=0$, that is when $M=0$, or

$$\tfrac{1}{2}wlx - \tfrac{1}{2}wx^2 - \tfrac{1}{12}wl^2 = 0,$$

from which $x=(\tfrac{1}{2} \pm \tfrac{1}{6}\sqrt{3})l = 0{\cdot}211l$ and $0{\cdot}789l$ approximately. These values give the positions of the points of inflexion, E and F, which are seen to be equidistant from the centre C.

Integrating the expression for $\dfrac{dy}{dx}$ gives

$$y = -\frac{w}{2EI}(\tfrac{1}{6}lx^3 - \tfrac{1}{12}x^4 - \tfrac{1}{12}l^2x^2 + B'),$$

where B' is a constant of integration. When $x=0$, $y=0$, therefore $B'=0$, and

$$y = -\frac{w}{12EI}(lx^3 - \tfrac{1}{2}x^4 - \tfrac{1}{2}l^2x^2),$$

which applies for values of x from 0 to l.

The maximum deflection occurs when $x=\frac{1}{2}l$, therefore

$$y_{max}=-\frac{w}{12EI}(\tfrac{1}{8}l^4-\tfrac{1}{32}l^4-\tfrac{1}{8}l^4)=\frac{wl^4}{384EI},$$

and if wl is denoted by W,

$$y_{max}=\frac{Wl^3}{384EI}.$$

At the point of inflexion E, $x=(\frac{1}{2}-\frac{1}{6}\sqrt{3})l$; substituting this value in the expression for y gives the deflection at E,

$$y_E=\frac{wl^4}{864EI}=\frac{Wl^3}{864EI},$$

which is also the deflection at the point F.

104. **Beam Fixed at One End, Supported at the Other End, and Loaded Uniformly.**—A beam AB, of length l between the supports, is fixed at B, simply supported at A, and carries a uniformly distributed load w per unit length (Fig. 182). The supports are at the same level and the slope at B is zero.

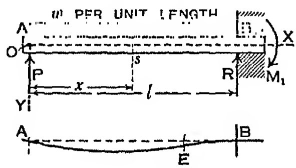

FIG. 182.

Let P and R be the reactions at A and B respectively, and let M_1 be the fixing moment at B. The point E in the lower part of the Fig. is the point of inflexion, the position of which is as yet unknown.

Equating vertical forces, $P+R=wl$, but since P and R are unequal their values cannot be determined until another equation has been obtained.

The origin may, of course, be at A or at B, and each of these ways of starting the analysis has its advantages. Taking the origin at A, then at a section s, a distance x from A, the bending moment is

$$M=Px-\tfrac{1}{2}wx^2,$$

and so $$\frac{d^2y}{dx^2}=-\frac{M}{EI}=-\frac{1}{EI}(Px-\tfrac{1}{2}wx^2) \quad . \quad . \quad (1).$$

Therefore $\frac{dy}{dx} = -\frac{1}{EI}(\frac{1}{2}Px^2 - \frac{1}{6}wx^3 + A')$ . . (2),

and $y = -\frac{1}{EI}(\frac{1}{6}Px^3 - \frac{1}{24}wx^4 + A'x + B')$. (3),

where A′ and B′ are constants of integration.

When $x=0$, $y=0$, therefore $B'=0$.

When $x=l$, $y=0$, therefore $0=\frac{1}{6}Pl^3 - \frac{1}{24}wl^4 + A'l$.

When $x=l$, $\frac{dy}{dx}=0$, therefore $0=\frac{1}{2}Pl^2 - \frac{1}{6}wl^3 + A'$.

Solving these two equations gives

$$P=\tfrac{3}{8}wl \quad \text{and} \quad A' = -\tfrac{1}{48}wl^3.$$

Then $R=wl-P=\frac{5}{8}wl$, and $M_1=\frac{1}{2}wl^2 - Pl = \frac{1}{8}wl^2$.

At E, the point of inflexion, $\frac{d^2y}{dx^2}=0$, therefore putting $P=\frac{3}{8}wl$ in (1) and equating to zero,

$$\tfrac{3}{8}wlx - \tfrac{1}{2}wx^2 = 0,$$

from which $x=0$ or $\frac{3}{4}l$, and the second of these values is the one required.

Putting $x=\frac{3}{4}l$, $P=\frac{3}{8}wl$, $A'=-\frac{1}{48}wl^3$, and $B'=0$ in (3) and simplifying, gives the deflection at E, that is

$$y_E = \frac{5wl^4}{2048EI} = \frac{5Wl^3}{2048EI}$$

where $W=wl$.

The position of the point of maximum deflection is found by putting $\frac{dy}{dx}=0$. Substituting the values of P and A′ in (2) and equating to zero,

$$\tfrac{3}{16}wlx^2 - \tfrac{1}{6}wx^3 - \tfrac{1}{48}wl^3 = 0.$$

The slope is zero when $x=l$, so $x=l$ is a root of this equation and $x-l$ is a factor. Dividing by $w(x-l)$ gives the other factor which is also zero, and the equation becomes

$$-\tfrac{1}{6}x^2 + \tfrac{1}{48}lx + \tfrac{1}{48}l^2 = 0$$

from which $x=\frac{1}{16}l(1\pm\sqrt{33})$, and the required root is

$$x=\tfrac{1}{16}l(1+\sqrt{33})=0{\cdot}4215l \quad \text{approximately,}$$

the other root being outside the range of the data.

Putting this value of x and also the values of P, A′, and B′ in (3) and simplifying, gives

$$y_{\max}=\frac{wl^4}{185\text{EI}}=\frac{\text{W}l^3}{185\text{EI}} \quad \text{nearly.}$$

105. Beam Supported at the Ends and Loaded at any Intermediate Point.—A beam AB of length l between the supports at A and B (Fig. 183) carries a load W at a point C which is at distances a and b from A and B respectively. Therefore $a+b=l$.

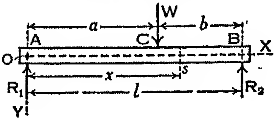

FIG. 183.

Denoting the reactions at A and B by R_1 and R_2 respectively, then

$$R_1=\frac{bW}{l} \quad \text{and} \quad R_2=\frac{aW}{l}.$$

Taking the origin at A, then at a section s, a distance x from A, the bending moment is

$$M=R_1x \qquad \text{if } x \text{ is less than } a,$$

and

$$M=R_1x-W(x-a) \quad \text{if } x \text{ is greater than } a.$$

Therefore, from A to C,

$$\frac{d^2y}{dx^2}=-\frac{1}{\text{EI}}\{R_1x\},$$

and from C to B,

$$\frac{d^2y}{dx^2}=-\frac{1}{\text{EI}}\{R_1x-W(x-a)\}.$$

By integrating each of these equations twice, four constants of integration are obtained. Now $y=0$ at A and at B, also each equation for y must give the same

deflection at C and each equation for $\frac{dy}{dx}$ must give the same slope at C. These conditions enable the four constants to be determined. If a beam carries more than one load, the work may be done in a similar way but it is laborious.

Instead of completing the problem in the way described, it will be solved by the *Macaulay method* in the next Art.

106. Macaulay Method for Obtaining Deflections when the Loading is Not Continuous.—The method will be explained by applying it to the beam supported and loaded as described in the preceding Art. and illustrated in Fig. 183. The general rules are given at the end of this Art.

With the origin at A, the bending moment at any section between C and B is

$$M = R_1 x - W[x-a] = W\left\{\frac{bx}{l} - [x-a]\right\},$$

and this is also the value of M when x is less than a if the term $-[x-a]$ is omitted. Now if x is less than a, then $x-a$ is negative. Therefore the equation gives the bending moment at any section of the beam *provided the term containing the square brackets* [ ] *is omitted when the expression inside these brackets becomes negative.*

With this proviso, the equation

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} = -\frac{W}{EI}\left\{\frac{bx}{l} - [x-a]\right\} \quad . \quad . \quad (1)$$

applies from $x=0$ to $x=l$.

Now it is possible to integrate $[x-a]$ with respect to x in two ways, either term by term or as a whole, and the two results differ only by a constant. When using the Macaulay method, terms such as $[x-a]$ are integrated as a whole; therefore, integrating (1),

$$\frac{dy}{dx} = -\frac{W}{EI}\left\{\frac{bx^2}{2l} - \frac{[x-a]^2}{2} + A'\right\} \quad . \quad . \quad (2),$$

where A' is a constant of integration, and this equation

applies to the whole beam provided the term containing the square brackets [ ] is omitted when the expression inside these brackets becomes negative.

It will now be shown that the value of the constant A′ is the same when x is less than a as when x is greater than a.

When x is less than a,

$$\frac{d^2y}{dx^2}=-\frac{W}{EI}\left\{\frac{bx}{l}\right\} \quad \text{and} \quad \frac{dy}{dx}=-\frac{W}{EI}\left\{\frac{bx^2}{2l}+A''\right\},$$

supposing the constant is now A″.

When $x=a$, $$\frac{dy}{dx}=-\frac{W}{EI}\left\{\frac{ba^2}{2l}+A''\right\},$$

but from (2), when $x=a$,

$$\frac{dy}{dx}=-\frac{W}{EI}\left\{\frac{ba^2}{2l}+A'\right\}.$$

Since these two results must be identical it follows that $A''=A'$.

Finally, integrating (2),

$$y=-\frac{W}{EI}\left\{\frac{bx^3}{6l}-\frac{[x-a]^3}{6}+A'x+B'\right\} \quad . \quad (3),$$

where B′ is a constant of integration, and this equation applies to the whole beam provided the term containing the square brackets [ ] is omitted when the expression inside these brackets becomes negative. It will be left to the student to show that the constant B′ has the same value whatever the value of x may be.

The constants A′ and B′ will now be determined. When $x=0$, then $y=0$, and $x-a$ is negative and therefore neglected, hence it follows from (3) that $B'=0$.

When $x=l=a+b$, $y=0$, therefore from (3)

$$0=\tfrac{1}{6}bl^2-\tfrac{1}{6}b^3+A'l,$$

from which

$$A'=-\frac{bl}{6}+\frac{b^3}{6l}=-\frac{ab}{6l}(a+2b).$$

Therefore

$$y = -\frac{W}{6EI}\left\{\frac{bx^3}{l} - [x-a]^3 - \frac{ab}{l}(a+2b)x\right\} \quad . \quad (4).$$

The deflection under the load, that is at the point C, is obtained by putting $x=a$. Denoting this deflection by y_C, then

$$y_C = -\frac{W}{6EIl}\{ba^3 - a^2b(a+2b)\} = \frac{Wa^2b^2}{3EI(a+b)}.$$

The maximum deflection is not at the point C, but at a point in the longer of the two parts AC and CB, and at this point the slope is zero. Assuming that AC is greater than CB, as in Fig. 183, then putting $\frac{dy}{dx}=0$ in equation (2), neglecting the term in square brackets because $x-a$ is negative, and substituting the value of A′,

$$0 = \frac{bx^2}{2l} - \frac{ab}{6l}(a+2b),$$

from which $\qquad x = \{\tfrac{1}{3}a(a+2b)\}^{\frac{1}{2}}.$

Substituting this value of x in (4), omitting the square-brackets term and simplifying, gives

$$y_{max} = \frac{Wb}{3EI(a+b)}\left\{\frac{a(a+2b)}{3}\right\}^{\frac{3}{2}}.$$

General Rules.—Express the bending moment at a suitable section so as to include all the loads. Put in square brackets the expressions which sometimes have to be neglected. Neglect any term in square brackets when the expression inside these brackets becomes negative. Integrate as a whole any term in square brackets.

107. Beam Fixed at One End, Supported at the Other End, and Loaded at the Centre.—A beam AB (Fig. 184), fixed at B and simply supported at A, carries a load W at the centre C. The length AB is l, the supports are at the same level, and the slope at B is zero.

Let P and R be the reactions at A and B respectively, and let M_1 be the fixing moment at B. Since M_1 is unknown, the values of the reactions cannot be obtained at once by equating moments.

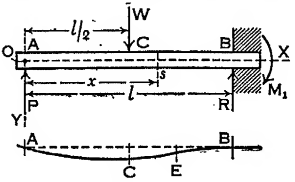

FIG. 184.

Using the Macaulay method, the required bending moment may be suitably expressed in four ways, two ways with the origin at A and two ways with the origin at B (see Ex. 8, p. 238). Taking the origin at A, then at a section s, between C and B and a distance x from A, the bending moment is $M = Px - W[x - \frac{1}{2}l]$.

Therefore

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} = -\frac{1}{EI}\{Px - W[x - \tfrac{1}{2}l]\} \quad . \quad . \quad (1),$$

$$\frac{dy}{dx} = -\frac{1}{EI}\{\tfrac{1}{2}Px^2 - \tfrac{1}{2}W[x - \tfrac{1}{2}l]^2 + A'\} \quad . \quad . \quad (2),$$

and

$$y = -\frac{1}{EI}\{\tfrac{1}{6}Px^3 - \tfrac{1}{6}W[x - \tfrac{1}{2}l]^3 + A'x + B'\} \quad . \quad (3),$$

where A′ and B′ are constants of integration.

[Two of the ways of stating the bending moment contain the expression $[\frac{1}{2}l - x]$. The student should note that $\int[\frac{1}{2}l - x]dx = -\frac{1}{2}[\frac{1}{2}l - x]^2 + \text{a const.}$ It is a common mistake to omit the minus sign in front of the bracket.]

The constants A′ and B′ will now be determined.

When $x = 0$, then $y = 0$, and substituting in (3) gives

$$B' = 0.$$

When $x = l$, then $y = 0$, and substituting in (3) gives

$$0 = \tfrac{1}{6}Pl^3 - \tfrac{1}{48}Wl^3 + A'l.$$

When $x = l$, then $\frac{dy}{dx} = 0$, and substituting in (2) gives

$$0 = \tfrac{1}{2}Pl^2 - \tfrac{1}{8}Wl^2 + A'.$$

Solving these two equations gives $P=\frac{5}{16}W$ and $A'=-\frac{1}{32}Wl^2$. Then

$$R=W-P=\tfrac{11}{16}W, \quad \text{and} \quad M_1=\tfrac{1}{2}Wl-Pl=\tfrac{3}{16}Wl.$$

At E, the point of inflexion (see lower part of Fig.), $\frac{d^2y}{dx^2}=0$, therefore from (1), putting $P=\frac{5}{16}W$,

$$0=\tfrac{5}{16}Wx-W[x-\tfrac{1}{2}l], \quad \text{from which} \quad x=\tfrac{8}{11}l.$$

The square-brackets term was not omitted, because it is obvious that E must be between the points C and B, for there is no bending moment at a point of inflexion and there is a bending moment at every point between A and C.

The deflection at E is found by putting $x=\frac{8}{11}l$ in (3) and also substituting for P, A′, and B′. The result, after simplification, is

$$y_E=\frac{9Wl^3}{1936EI}.$$

Similarly, the deflection under the load at C, where $x=\frac{1}{2}l$, is

$$y_C=\frac{7Wl^3}{768EI}.$$

The maximum deflection occurs at a point between A and C; this can be provéd by putting $x=\frac{1}{2}l$ in (2) and showing that the slope is negative at C. Then, since the slope is positive when $x=0$, it follows that it is zero at a point between A and C.

Putting $\frac{dy}{dx}=0$ in (2) and substituting the values of P and A′, omitting the square-brackets term,

$$0=\tfrac{1}{2}\cdot\tfrac{5}{16}Wx^2-\tfrac{1}{32}Wl^2, \quad \text{from which} \quad x=\frac{l}{\sqrt{5}}.$$

Putting this value of x and the values of P, A′, and B′ in (3), omitting the square-brackets term, and simplifying, gives

$$y_{max}=\frac{Wl^3}{48\sqrt{5}EI}=\frac{Wl^3}{107EI} \quad \text{nearly.}$$

P

108. Beam Supported at each End and Carrying Several Concentrated Loads.—A beam AB, of length l between the supports at A and B (Fig. 185), carries loads W_1, W_2, and W_3 at distances a, b, and c, respectively, from A.

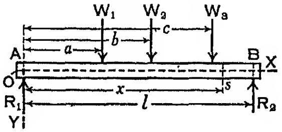

FIG. 185.

Let R_1 and R_2 be the reactions at A and B respectively. Taking the origin at A and considering a section s between the load W_3 and the end B, the bending moment is

$$M = R_1 x - W_1[x-a] - W_2[x-b] - W_3[x-c].$$

Putting $M=0$ and $x=l$ and solving for R_1 gives

$$R_1 = \frac{1}{l}\{W_1[l-a] + W_2[l-b] + W_3[l-c]\},$$

then $$R_2 = W_1 + W_2 + W_3 - R_1.$$

Now $$\frac{d^2y}{dx^2} = -\frac{M}{EI} = -\frac{1}{EI}\{R_1 x - W_1[x-a] - W_2[x-b] - W_3[x-c]\} \quad (1).$$

Integrating, $$\frac{dy}{dx} = -\frac{1}{EI}\{\tfrac{1}{2}R_1 x^2 - \tfrac{1}{2}W_1[x-a]^2 - \tfrac{1}{2}W_2[x-b]^2 - \tfrac{1}{2}W_3[x-c]^2 + A'\} \quad (2),$$

and $$y = -\frac{1}{EI}\{\tfrac{1}{6}R_1 x^3 - \tfrac{1}{6}W_1[x-a]^3 - \tfrac{1}{6}W_2[x-b]^3 - \tfrac{1}{6}W_3[x-c]^3 + A'x + B'\} \quad (3),$$

where A' and B' are constants of integration.

When $x=0$, $y=0$, therefore $B'=0$ since all the expressions in square brackets become negative and are neglected.

When $x=l$, $y=0$, therefore

$$0 = \tfrac{1}{6}R_1 l^3 - \tfrac{1}{6}W_1[l-a]^3 - \tfrac{1}{6}W_2[l-b]^3 - \tfrac{1}{6}W_3[l-c]^3 + A'l,$$

from which

$$A' = -\frac{R_1 l^2}{6} + \frac{1}{6l}\{W_1[l-a]^3 + W_2[l-b]^3 + W_3[l-c]^3\}.$$

The deflections at the points under the loads W_1, W_2, and W_3 are found by giving x the values a, b, and c, respectively, in equation (3). Denoting these deflections by y_1, y_2, and y_3, respectively, then

$$y_1 = -\frac{1}{6EI}\{R_1a^3 + 6A'a\},$$

$$y_2 = -\frac{1}{6EI}\{R_1b^3 - W_1[b-a]^3 + 6A'b\},$$

$$y_3 = -\frac{1}{6EI}\{R_1c^3 - W_1[c-a]^3 - W_2[c-b]^3 + 6A'c\},$$

and the values of R_1 and A' have to be substituted in these equations.

If the numerical values of the loads and of a, b, c, and l are known, the position of the section at which the deflection is a maximum can be determined. The sign of the slope is examined at the end A and at the point under each load in turn, and, if necessary, at the end B; the sign will change between two of these points and consequently the slope will be zero at an intermediate point. Then this point may be found by putting $\frac{dy}{dx}=0$ in equation (2), omitting the unwanted square-brackets term or terms, and solving for x. Finally, substituting this value of x in equation (3) will give the maximum deflection.

109. A Numerical Example with a Propped Beam.—A beam, fixed in direction at each end and resting on a prop at an intermediate point, is loaded as shown in Fig. 186. (The loads are in tons and the lengths are in feet.) The three supports are at the same level and the slope is zero at each end.

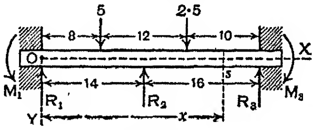

Fig. 186.

It is required to find the values of the reactions R_1, R_2, and R_3, and of the fixing moments M_1 and M_3. Also, given

that $E = 30 \times 10^6$ lb./in.^{2} and $I = 55\cdot63$ in.^{4}, it is required to find the maximum deflection in each of the two portions into which the beam is divided by the prop.

It will be seen that the Macaulay method may still be applied although the beam has an intermediate support. This support introduces an extra unknown force and so increases the algebraic and numerical work, but apart from this there is no difficulty.

Taking the origin on the left and considering a section s to the right of the loads, then

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} = -\frac{1}{EI}\{R_1x + R_2[x-14] - 5[x-8] - 2\cdot5[x-20] - M_1\} \quad (1),$$

$$\frac{dy}{dx} = -\frac{1}{EI}\{\tfrac{1}{2}R_1x^2 + \tfrac{1}{2}R_2[x-14]^2 - \tfrac{5}{2}[x-8]^2 - \tfrac{5}{4}[x-20]^2 - M_1x + A\} \quad (2),$$

$$y = -\frac{1}{EI}\{\tfrac{1}{6}R_1x^3 + \tfrac{1}{6}R_2[x-14]^3 - \tfrac{5}{6}[x-8]^3 - \tfrac{5}{12}[x-20]^3 - \tfrac{1}{2}M_1x^2 + Ax + B\} \quad (3),$$

where A and B are constants of integration.

When $x = 0$, $\frac{dy}{dx} = 0$, therefore $A = 0$.

When $x = 30$, $\frac{dy}{dx} = 0$, therefore

$$\tfrac{1}{2} \times 30^2R_1 + \tfrac{1}{2} \times 16^2R_2 - \tfrac{5}{2} \times 22^2 - \tfrac{5}{4} \times 10^2 - 30M_1 = 0 \quad (4).$$

When $x = 0$, $y = 0$, therefore $B = 0$.

When $x = 14$, $y = 0$, therefore

$$\tfrac{1}{6} \times 14^3R_1 - \tfrac{5}{6} \times 6^3 - \tfrac{1}{2} \times 14^2M_1 = 0 \quad . \quad (5).$$

When $x = 30$, $y = 0$, therefore

$$\tfrac{1}{6} \times 30^3R_1 + \tfrac{1}{6} \times 16^3R_2 - \tfrac{5}{6} \times 22^3 - \tfrac{5}{12} \times 10^3 - \tfrac{1}{2} \times 30^2M_1 = 0 \quad (6).$$

Solving the simultaneous equations (4), (5), and (6) gives

$M_1 = 8\cdot397$ ton-ft., $R_1 = 2\cdot193$ tons, and $R_2 = 4\cdot688$ tons.

Then

$R_3 = \text{Total load} - R_1 - R_2 = 7\cdot5 - 2\cdot193 - 4\cdot688 = 0\cdot619$ ton.

Taking moments about the right-hand end,

$$30R_1+16R_2+M_3-5\times 22-2{\cdot}5\times 10-M_1=0.$$

Substituting the known values and solving, gives

$$M_3=2{\cdot}599 \text{ ton-ft.}$$

Before the points of maximum deflection can be found, it is necessary to examine the sign of the slope of the beam at each load, in order to know the terms to neglect when equating the slope to zero. As already shown, the constant A is zero.

At the 5-ton load, $x=8$. Putting this value in (2),

$$\frac{dy}{dx}=-\frac{1}{EI}\{\tfrac{1}{2}\times 8^2R_1-8M_1\}=-\frac{1}{EI}\{70{\cdot}18-67{\cdot}18\},$$

which is negative; but near the origin the slope is positive, therefore a point of maximum deflection occurs between $x=0$ and $x=8$.

From (2), putting $\frac{dy}{dx}=0$, neglecting all the square-brackets terms because x is less than 8, $\frac{1}{2}R_1x^2-M_1x=0$.

Hence $x=0$, or $x=\frac{2M_1}{R_1}=\frac{2\times 8{\cdot}397}{2{\cdot}193}=7{\cdot}66$, and substituting the latter value in (3) gives the first maximum deflection. As already shown, the constants A and B are zero.

$$\textit{First } y_{max}=-\frac{1}{EI}\{\tfrac{1}{6}R_1\times 7{\cdot}66^3-\tfrac{1}{2}M_1\times 7{\cdot}66^2\}$$

$$=-\frac{1}{EI}\{\tfrac{1}{6}\times 2{\cdot}193\times 7{\cdot}66^3-\tfrac{1}{2}\times 8{\cdot}397\times 7{\cdot}66^2\}$$

$$=\frac{82{\cdot}1}{EI},$$

and if E and I are in ton and foot units, the deflection will be in feet.

Substituting the given values of E and I,

$$\textit{First } y_{max}=82{\cdot}1\times\frac{2240}{30\times 10^6\times 12^2}\times\frac{12^4}{55{\cdot}63}\times 12$$

$$=0{\cdot}190 \text{ inch.}$$

At the 2·5 ton load, $x=20$. Putting this value in (2),

$$\frac{dy}{dx}=-\frac{1}{EI}\{\tfrac{1}{2}\times 20^2R_1+\tfrac{1}{2}\times 6^2R_2-\tfrac{5}{2}\times 12^2-20M_1\}$$

$$=-\frac{1}{EI}\{438{\cdot}6+84{\cdot}4-360-167{\cdot}9\}=-\frac{1}{EI}\{523{\cdot}0-527{\cdot}9\}$$

which is positive; but when x is slightly less than 30 the slope is negative, therefore a point of maximum deflection occurs between $x=20$ and $x=30$.

From (2), putting $\frac{dy}{dx}=0$, including all the square-brackets terms since x is greater than 20,

$$\tfrac{1}{2}R_1x^2+\tfrac{1}{2}R_2[x-14]^2-\tfrac{5}{6}[x-8]^2-\tfrac{5}{4}[x-20]^2-M_1x=0.$$

Substituting the known values and solving for x, gives $x=21{\cdot}6$ or $30{\cdot}0$, then if $x=21{\cdot}6$ is substituted in (3) the second maximum deflection is obtained.

Having obtained the information that the second point of maximum deflection lies between $x=20$ and $x=30$, the subsequent arithmetical work may be shortened considerably by taking the origin on the right. Denoting $30-x$ by X, then from $X=0$ to $X=10$,

$$\frac{d^2y}{dX^2}=-\frac{1}{EI}\{R_3X-M_3\} \quad . \quad . \quad (7).$$

The integration constants will be zero, as before, therefore

$$\frac{dy}{dX}=-\frac{1}{EI}\{\tfrac{1}{2}R_3X^2-M_3X\} \quad . \quad . \quad (8),$$

and

$$y=-\frac{1}{EI}\{\tfrac{1}{6}R_3X^3-\tfrac{1}{2}M_3X^2\} \quad . \quad . \quad (9).$$

Putting $\frac{dy}{dX}=0$, $\tfrac{1}{2}R_3X^2-M_3X=0$,

from which

$$X=0, \quad \text{or} \quad X=\frac{2M_3}{R_3}=\frac{2\times 2{\cdot}599}{0{\cdot}619}=8{\cdot}40,$$

and substituting the latter value of X and the values of R_3 and M_3 in (9), gives

$$Second\ y_{max} = -\frac{1}{EI}\{\tfrac{1}{6} \times 0{\cdot}619 \times 8{\cdot}40^3 - \tfrac{1}{2} \times 2{\cdot}599 \times 8{\cdot}40^2\}$$

$$= \frac{30{\cdot}5}{EI},$$

and if E and I are in ton and foot units, the deflection will be in feet.

Substituting the given values of E and I,

$$Second\ y_{max} = 30{\cdot}5 \times \frac{2240}{30 \times 10^6 \times 12^2} \times \frac{12^4}{55{\cdot}63} \times 12$$

$$= 0{\cdot}071 \text{ inch.}$$

110. Beam with a Load Uniformly Distributed over Part of its Length.—When a load is uniformly distributed over a part of a beam, the deflection problem may be solved by the Macaulay method without introducing any new idea, provided the loading extends to one end. If, however, the loading does not extend to one end, then an artifice is needed and this will be explained by considering an example.

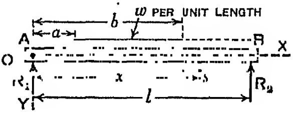

FIG. 187.

A beam AB is supported at A and B and the distance between the supports is l (Fig. 187). A load w per unit length is applied over a length $b - a$, the ends of the load being at distances a and b from A.

Let R_1 and R_2 be the reactions as shown. Taking moments about A,

$$R_2 l = w(b-a)\frac{b+a}{2}, \quad \text{therefore} \quad R_2 = \frac{w(b^2-a^2)}{2l}.$$

$$\text{Then} \quad R_1 = w(b-a) - R_2 = w(b-a)\left\{1 - \frac{b+a}{2l}\right\}.$$

The artifice now required is as follows. Taking the origin at A, imagine the load w per unit length to be

continued to the end B, that is over the length $l-b$, then suppose there is a similar equal load acting upwards over the same length. These imaginary loads are shown dotted in the Fig.

At a section s in the part of the beam where the imaginary loads are placed and a distance x from the origin, the bending moment is

$$M = R_1x - \tfrac{1}{2}w[x-a]^2 + \tfrac{1}{2}w[x-b]^2,$$

therefore

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} = -\frac{1}{EI}\{R_1x - \tfrac{1}{2}w[x-a]^2 + \tfrac{1}{2}w[x-b]^2\},$$

and the rest of the work is done in the usual way.

111. **Beam Acted on by a Couple at an Intermediate Section.**—A beam AB of length l has a couple M_C applied to it at an intermediate section C, a distance a from A (Fig. 188). The couple is shown as tending to turn the beam in the plane of the paper in an anticlockwise direction, and the reactions R_1 and R_2 are applied as indicated to give equilibrium.

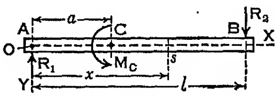

Fig. 188.

Taking moments about B,

$$R_1l - M_C = 0, \quad \text{therefore} \quad R_1 = M_C/l.$$

Taking moments about A,

$$R_2l - M_C = 0, \quad \text{therefore} \quad R_2 = M_C/l.$$

With the origin at A, then at a section s between C and B, the bending moment is $M = R_1x - M_C$, therefore

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} = -\frac{1}{EI}\{R_1x - M_C\} \quad . \quad . \quad (1),$$

and this equation holds from $x=a$ to $x=l$; when x is less than a the couple M_C must be omitted.

By introducing an artifice in the first integration, the equations of slope and deflection are made so that they

may be applied to the whole beam. This artifice is an extension of the Macaulay method and is due to H. A. Webb.

Integrating (1), writing $M_0[x-a]$ instead of M_0x, then

$$\frac{dy}{dx}=-\frac{1}{EI}\{\tfrac{1}{2}R_1x^2-M_0[x-a]+A'\} \quad . \quad . \quad (2),$$

and
$$y=-\frac{1}{EI}\{\tfrac{1}{6}R_1x^3-\tfrac{1}{2}M_0[x-a]^2+A'x+B'\} \quad . \quad (3),$$

where A' and B' are constants of integration, and the square-brackets terms are neglected as usual when x is less than a. The additional constant term M_0a in (2) is allowed for automatically when the value of the constant A' is determined.

From the conditions $y=0$ when $x=0$ and when $x=l$, the values of A' and B' may be obtained, then the slope and deflection may be found at any point on the beam.

112. Frames with Rigid Joints.—In a number of simple cases the problems concerning frames which have rigid joints may be solved by considering each member of the frame as a beam. When two members are rigidly connected, then for equilibrium they exert equal bending moments at the joint, the direction being clockwise on one member and anticlockwise on the other member; also the angular distortion at the joint is the same for each member. As is usual in beam problems, it is assumed that the distance between the two ends of a member is not altered by the bending of the member.

The following example illustrates the general procedure.

Example.—A load W is supported by a ⊔-shaped framework hung on pins passing through holes at A and B in the manner shown in Fig. 189. The sections of the vertical and horizontal parts of the bracket are the same and their effective lengths may be taken as h and l respectively. If the distance between the pins is unchanged by the

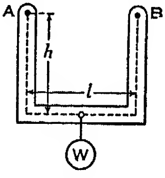

FIG. 189.

application of the load and the corners remain right-angled, prove that at these corners the horizontal and vertical members are subjected to bending moments of magnitude

$$\frac{3}{8} \cdot \frac{l^2}{2h+3l} \cdot \mathrm{W}. \qquad \text{[C.U.]}$$

An exaggerated idea of the way in which the framework deflects is given in Fig. 190. Also, the left-hand vertical member and the horizontal member are shown separately with the forces and couples required for equilibrium. The right-hand member is loaded in a similar manner to the left-hand member.

If a force or a couple were shown acting in the wrong sense, then its numerical value would be found to be negative.

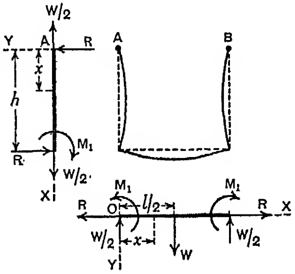

FIG. 190.

The horizontal member and the left-hand vertical member will now be considered in turn, and the bending moment at the joint will be found by equating two expressions for the slope at the joint.

Horizontal Member.—Each vertical reaction is $\frac{1}{2}$W and the unknown bending moment at each end will be denoted by M_1. There is a longitudinal pull R, but the effect of this on the bending moment at any section may be neglected.

Taking the origin at the left-hand end, then, provided x is less than $\frac{1}{2}l$,

$$\frac{d^2y}{dx^2} = -\frac{\mathrm{M}}{\mathrm{EI}} = -\frac{1}{\mathrm{EI}}\{\tfrac{1}{2}\mathrm{W}x - \mathrm{M}_1\},$$

and
$$\frac{dy}{dx} = -\frac{1}{\mathrm{EI}}\{\tfrac{1}{4}\mathrm{W}x^2 - \mathrm{M}_1 x + \mathrm{A}'\},$$

where A′ is a constant of integration. When $x = \frac{1}{2}l$, the

slope is zero, therefore

$$0=\tfrac{1}{16}Wl^2-\tfrac{1}{2}M_1l+A' \quad\text{and}\quad A'=-\tfrac{1}{16}Wl^2+\tfrac{1}{2}M_1l.$$

When $x=0$,

$$\frac{dy}{dx}=-\frac{1}{EI}\{A'\}=\frac{1}{EI}\{\tfrac{1}{16}Wl^2-\tfrac{1}{2}M_1l\} \quad . \quad (1).$$

Left-hand Vertical Member.—The tension is $\frac{1}{2}W$, but its effect on the bending moment at any section may be neglected. The couple at the lower end is M_1 and this is balanced by the reactions R placed as shown.

Taking moments about A, $Rh-M_1=0$, therefore $Rh=M_1$.

With the origin at A, then at a section a distance x from A,

$$\frac{d^2y}{dx^2}=-\frac{M}{EI}=-\frac{1}{EI}\{-Rx\},$$

$$\frac{dy}{dx}=\frac{1}{EI}\{\tfrac{1}{2}Rx^2+A''\},$$

and

$$y=\frac{1}{EI}\{\tfrac{1}{6}Rx^3+A''x+B''\},$$

where A'' and B'' are constants of integration.

When $x=0$, $y=0$, therefore $B''=0$.

When $x=h$, $y=0$ (neglecting the extension of the horizontal member due to the pull R), therefore

$$0=\tfrac{1}{6}Rh^3+A''h, \quad\text{from which}\quad A''=-\tfrac{1}{6}Rh^2.$$

Therefore $$\frac{dy}{dx}=\frac{1}{EI}\{\tfrac{1}{2}Rx^2-\tfrac{1}{6}Rh^2\}.$$

When $x=h$,

$$\frac{dy}{dx}=\frac{1}{EI}\{\tfrac{1}{2}Rh^2-\tfrac{1}{6}Rh^2\}=\frac{1}{EI}\{\tfrac{1}{3}M_1h\} \quad . \quad (2),$$

remembering that $Rh=M_1$.

The slopes given by (1) and (2) are equal, therefore

$$\tfrac{1}{16}Wl^2-\tfrac{1}{2}M_1l=\tfrac{1}{3}M_1h, \quad\text{from which}\quad M_1=\frac{3Wl^2}{8(2h+3l)}$$

and this is the required bending moment.

113. Beams with Varying Sections.—If the section of a beam varies from point to point along the beam, then

$$\frac{d^2y}{dx^2} = -\frac{M}{EI}, \quad \frac{dy}{dx} = -\frac{1}{E}\int\frac{M}{I}dx, \quad \text{and} \quad y = -\frac{1}{E}\iint\frac{M}{I}dx\,dx:$$

A constant must, of course, be introduced at each integration. The chief point to notice is that I is not a constant and cannot be taken outside the integration symbol. In cases where it is possible to express both M and I in terms of x the integration may generally be done analytically (see Ex. 21, p. 242), but in other cases it must be done graphically.

The section of a beam may change abruptly at several points, so that each portion of the beam has a different uniform section and a different constant moment of inertia of section. As an example, consider a beam (Fig. 191) having plates riveted to the top and bottom flanges and loaded as shown. The section changes abruptly at B and C, and each of the portions AB, BC, and CD has its own constant moment of inertia of section.

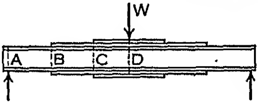

Fig. 191.

Three equations may be written down, one for each of the uniform portions AB, BC, and CD. Integrating each equation twice would give six constants of integration, and therefore six conditions are needed.

The conditions are as follows. The deflection is zero at A and the slope is zero at D. The equations for AB and BC give the same slope at B and the same deflection at B. The equations for BC and CD give the same slope at C and the same deflection at C.

These six conditions enable the constants to be determined, and then the deflection at any section may be calculated.

The work is somewhat laborious, and in complicated problems it saves time to plot M/I against x and then to integrate twice graphically. The deflection curve may also be obtained from the M/I curve by regarding the latter

as a load curve and then drawing a force polygon and a funicular polygon. In the graphical methods the constant E is taken into account when the scales of the diagrams are considered.

Exercises XI

When required assume $E = 30 \times 10^6$ *lb./in.*^{2} *for steel.*

1. A couple M_1 is applied to the free end of a cantilever of length l. Find (*a*) the maximum slope and (*b*) the maximum deflection.

2. A couple M_1 is applied at each end of a beam of length l so that it is in equilibrium. Find (*a*) the maximum slope and (*b*) the maximum deflection.

3. Two steel bars AB and BC, of circular cross-section and 2 inches diameter, are arranged as shown in Fig. 192 with AB vertical and BC horizontal, the joint at B being rigid. AB $=5$ feet and BC $=2$ feet. If a load $W=400$ lb. is applied at the end C, find the vertical and horizontal movements of C, neglecting the weight of the steel and the shortening of AB due to compression.

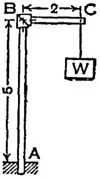

Fig. 192.

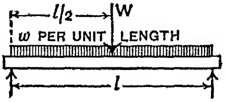

Fig. 193.

4. A beam of length l, freely supported at each end, carries a concentrated load W at the centre and a uniformly distributed load w per unit length (Fig. 193). Working from first principles, find the slope at each end and the central deflection.

5. A beam of length l carries a uniformly distributed load w per unit length. One end is fixed so that its slope is zero and the other end is freely supported so that it is d units lower than the fixed end. Find (*a*) the reaction at the fixed end, (*b*) the reaction at the freely supported end, and (*c*) the hogging bending moment at the fixed end.

6. A horizontal shaft of length l is subjected at its centre to a vertical load W. The shaft fits in bearings at its ends, and when

the slope of the shaft at the ends is θ the bearings exert a bending moment on the shaft of magnitude $\kappa\theta$. Prove that the central deflection of the shaft is

$$\frac{Wl^3}{192EI}\left(\frac{\kappa l+8EI}{\kappa l+2EI}\right). \qquad \text{[C.U.]}$$

7. A beam of uniform section and of length $2l$ is supported freely at its ends A and B and also at its centre C. The supports at A and B are rigid, but the support at C deflects a depth which is μ times the load it carries. If the beam is subjected to a total load W uniformly distributed along its whole length, show that the load on the central support is

$$\frac{5}{8}\cdot\frac{W}{1+\frac{6EI\mu}{l^3}}. \qquad \text{[C.U.]}$$

8. Refer to Art. 107, p. 223, where it is stated that the required bending moment may be suitably expressed in four ways. Write down the bending moment in these four ways.

9. The steel beam AB (Fig. 194) is freely supported at each end and loaded as shown. Find the values of the reactions R_1 and R_2 and the distance x from A of the section at which the maximum deflection occurs. Given that $I=377{\cdot}1$ inch^4, calculate the value of the maximum deflection in inches.

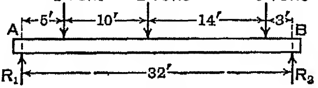

FIG. 194.

10. A beam AB of length $4a$ (Fig. 195) is freely supported at A and B. A load W is uniformly distributed over a length a between sections C and D. The positions of these sections are as shown: $AC=2a$, $CD=DB=a$. Find the distance from A of the section at which the maximum deflection occurs and the value of this deflection.

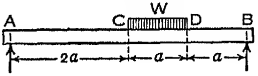

FIG. 195.

11. A straight bar AB (Fig. 196), of uniform cross-section and length $2l$, weighing w per unit length, is supported at points C and D which are at the same level. Find the positions of C and D in order that the slopes at A and B may be zero. Also find the deflections at the centre and the ends.

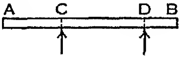

FIG. 196.

12. A beam AB of length l is freely supported at each end and carries loads W_1 and W_2 at points C and D which are at distances a_1 and a_2 from A and B respectively, as shown in Fig. 197. Show that the deflections at C and D are, respectively,

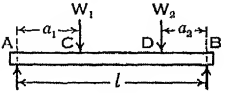

Fig. 197.

$$y_C = \frac{a_1}{6EIl}\{2a_1 W_1(l - a_1)^2 + a_2 W_2(l^2 - a_1^2 - a_2^2)\}$$

and $$y_D = \frac{a_2}{6EIl}\{2a_2 W_2(l - a_2)^2 + a_1 W_1(l^2 - a_1^2 - a_2^2)\}.$$

(These results are used in an example on whirling shafts, p. 194.)

Further, if $a_1 = a_2 = a$ and $W_1 = W_2 = W$, show that

$$y_C = y_D = \frac{Wa^2}{6EI}(3l - 4a).$$

13. The floor of the top storey of a warehouse is carried on steel joists freely supported at the ends. Each joist has to carry a load W which may be considered uniformly distributed. The central deflection is found to be excessive and, in order to reduce this, vertical tie rods are attached to the roof and to the joists, at points distant $l/3$ from each end, l being the length of a joist. If the lengths of these tie rods are adjusted so that the original central deflection is reduced by one-half, show that the bending moment at the middle of each joist is reduced to $\frac{47}{768}Wl$. [C.U.]

14. Fig. 198 illustrates a plate girder swing-bridge which can rotate about a central pivot C. The girder is of uniform section and the total dead-load weight is W.

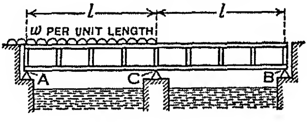

Fig. 198.

After being swung into position across the gap, the ends A and B are wedged up until the wedges exert an upward thrust R at each end. Prove that the deflection of the ends relatively to the centre is

$$\left(\frac{W}{16} - \frac{R}{3}\right)\frac{l^3}{EI}.$$

A live load w per unit length then advances across the bridge. Consider the position of this load as shown in Fig. 198. Prove

that in order to keep the end B from rising off its abutment the upward thrust R originally exerted by the wedges must not be less than $\frac{wl}{16}$. [C.U.]

15. A load W and a couple M_0 are applied at the centre of a beam of length l which is fixed at each end so that the end slopes are zero. The couple tends to bend the longitudinal axis of the beam in a vertical plane. Find the slope and the deflection at the centre of the beam.

16. Fig. 199 represents in skeleton a beam ACB encastered horizontally at A and B and supported on a column at its middle point C. This column is encastered at D and rigidly attached to the beam at C. The beam carries a load w per unit length extending from C to B, so that the deformations of the beam and column have the character indicated on the diagram, the points A, C, B being at the same level.

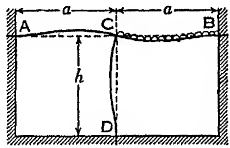

FIG. 199.

If I_1 and I_2 are the moments of inertia of the cross-sections of the beam and column respectively, prove that the bending moment in the column at C has the magnitude

$$\frac{wa^2}{12} \cdot \frac{1}{\left(1 + \frac{2h}{a} \cdot \frac{I_1}{I_2}\right)}. \qquad \text{[C.U.]}$$

17. The centre line of a thin-walled tube of circular section forms three sides of a rectangle in which $AB = \frac{1}{2}BC = CD = a$. The tube is fixed in a horizontal plane, its ends A and D being encastered into a vertical wall, and it supports a weight W at the middle of BC. Assuming that the modulus of rigidity for the material of the tube is $\frac{2}{5}$ its modulus of elasticity, show that the torque exerted on the portions AB and DC is $\frac{Wa}{9}$, and that the deflection of the weight is $\frac{5}{18}\frac{Wa^3}{EI}$, where I is the moment of inertia of the cross-section of the tube about its diameter. [C.U.]

18. A, B, C, D are four rigid supports at the same level, and the gaps between them are bridged by three discontinuous girders AB, BC, CD.

AB and CD have the same length a and the same moment of inertia I_a.

BC is of length b and its moment of inertia is I_b.

AB and CD each carry a uniformly distributed load w_a per unit length.

BC carries a uniformly distributed load w_b per unit length.

Show that these loaded girders can be lined up into one continuous length by applying to the ends of the girders at B and at C couples of magnitude

$$\frac{1}{4}\cdot\frac{w_a a^3 I_b + w_b b^3 I_a}{2aI_b + 3bI_a}. \qquad \text{[C.U.]}$$

19. A floor is carried by a series of discontinuous joists supported by stanchions. In order to strengthen the floor, double cantilever arms are inserted above the stanchions, as shown in Fig. 200, so that the joists have bearings at their quarter and

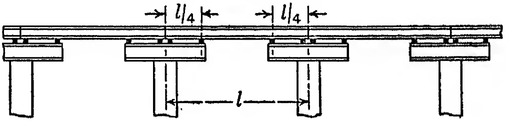

Fig. 200.

three-quarter points in addition to their ends. The joists are of uniform cross-section with moment of inertia I_1, and carry a uniformly distributed load w per unit length. The cantilever arms are also of uniform cross-section with moment of inertia I_2.

Show that the reduction in bending moment at the centre of a joist is

$$\frac{57I_2}{128(I_1+4I_2)}wl^2.$$

Also, by considering the bending moments at the centre and at the quarter point, show that the greatest reduction in bending moment for the joists is when $I_2 = 14I_1$ and that the maximum bending moment is then $\frac{wl^2}{64}$. [C.U.]

20. The bracket shown in Fig. 201 is built up of two members AD and BC, each five feet long, pin jointed at A and B and rigidly attached at D and C to a third member three feet long. It carries a point load of five tons at three feet from A. Assuming the moments of inertia of AD and BC to be double that of CD, determine the maximum positive and negative

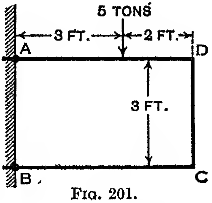

Fig. 201.

bending moments in each member and sketch the bending moment diagrams. [U.L.]

21. A flat plate spring of uniform thickness t is fixed horizontally at one end and loaded at the other end with a weight W. The plan of the spring is shown in Fig. 202.

Find an expression for the deflection of the free end; and prove that if c is small compared with b the expression reduces to that for a cantilever of uniform rectangular cross-section. [C.U.]

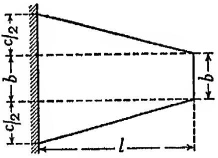

Fig. 202.

22. A horizontal steel beam, freely supported on a span of 20 feet, consists of a 16-inch × 6-inch at 50 lb. per foot rolled steel joist having a plate 10 inches wide by one-half an inch thick secured to each flange over the centre 10 feet of span. Determine the maximum deflection of the beam when completely loaded with two tons per foot run. Neglect rivet holes and the weight of the beam itself.

The moment of inertia of a 16-inch × 6-inch at 50 lb. per foot rolled steel joist alone is 618·1 inch units. [U.L.]

23. A simply supported beam of circular section and length l carries a concentrated load of W at mid-span. If the diameter of the section is the same for a length $\frac{l}{3}$ from each support, and is increased by 50 per cent. for the middle third of the span, determine the value of the constant in the formula for the deflection underneath the load, *deflection* $=$ *constant* $\times \frac{Wl^3}{EI}$, in which I is the moment of inertia of the section of the lesser diameter. [U.L.]

ANSWERS

Exercises II, pp. 28–31.

1. 21·3 ft./sec. at 59° 3′ to OX. 2. 59·5 ft./sec. at 40° 1′ to OX.
3. 6·09, 8·34, 9·74, and 10 ft./sec.
4. (*a*) At A; (*b*) At infinity perpendicular to AC.
5. 30·1 ft./sec. 6. Crankshaft 13·1 ft./sec., crank pin 10·8 ft./sec.
7. (*a*) At point of contact with rail; top point moves at 160 ft./sec.; (*b*) At centre of wheel; (*c*) On vertical diameter and $1\frac{3}{4}$ ft. above rail; sliding velocity 7 ft./sec.; velocity of top point 17 ft./sec.; (*d*) 13 ft./sec. perpendicular to line joining foremost point to instantaneous centre.
8. (*a*) 37° 53′; (*b*) 55·71 ft./sec.; (*c*) 15·01 ft./sec.; (*d*) 73° 48′.
9. Vel. of P = 1·73 ft./sec.; vel. of D = 3·19 ft./sec.
10. $\omega_{BC} = 0{\cdot}36$ rad./sec.; $\omega_{CD} = 1{\cdot}27$ rad./sec.; $v_C = 1{\cdot}49$ ft./sec.; vel. of mid point of BC = 1·56 ft./sec.
11. $v_A = 7{\cdot}72$ ft./sec.; $v_F = 10{\cdot}1$ ft./sec.; $v_H = 5{\cdot}20$ ft./sec.
12. 2·06 ft./sec. 13. 10·2 ft./sec.

Exercises III, pp. 42–46.

1. 55 ft./sec.; 26 ft./sec.^{2}. 2. 20·6 ft./sec.; 241·2 ft.
3. 0·966 sec.; −37·1 ft./sec.^{2}. 5. 13° 18′ to OX; 680·6 ft.
6. 10 min. 7. 2·75 ft./sec.^{2}; 3·14 ft./sec.^{2}; 49·09 sec.
8. 72 ft./sec.; 3240 ft. 9. 18·20 ft./sec.; 22·36 ft./sec.
10. 33·7 m.p.h.; 72·2 m.p.h.; 0·416 ft./sec.^{2}.
12. 0·702 ft./sec.^{2}; 63·51 ft./sec. 13. 31·1 sec.; 26,900 ft.; 3880 ft.
14. 31·5 sec.; 27,300 ft. 15. 26° 45′; 68° 1′; 59·5 ft.; 10·5 ft.
16. 3·05 sec. 18. 0, 29·6, 36·1, 21·5, 0 ft./sec.
19. 6958, 4013, −1283, −4013, −4392 ft./sec.^{2}. 20. 79°.
24. $\dfrac{\sqrt{2}\sin\theta}{(3-2\sqrt{2}\cos\theta)^{\frac{1}{2}}}\mathrm{AC}\,.\,\omega$; $\dfrac{-\sqrt{2}\sin\theta}{(3-2\sqrt{2}\cos\theta)^{2}}\omega^2$.

Exercises IV, pp. 60–67.

An acceleration is given here as negative when the velocity is decreasing.

1. 6480 ft./sec.^{2}. 2. (*a*) $\omega^2 r\left(1+\dfrac{r}{l}\right)$ ft./sec.^{2}; (*b*) $-\omega^2 r\left(1-\dfrac{r}{l}\right)$ ft./sec.^{2}.
3. 124 ft./sec.^{2}. 4. (*a*) 371 ft./sec.^{2}; (*b*) −215 ft./sec.^{2}.
8. $\omega = 3{\cdot}76$ rad./sec.; $\dot{\omega} = 24{\cdot}14$ rad./sec.^{2}; 3·36 ft./sec. at 121° measured anticlockwise from BA.
10. Vel. of C = 44·2 ft./sec.; vel. of E = 25·6 ft./sec.; acc. of C = 11,300 ft./sec.^{2}; acc. of E = −15,600 ft./sec.^{2}; ang. vel. of BC = 79·6 rad./sec.; ang. vel. of DE = 102 rad./sec.; ang. acc. of BC = −15,100 rad./sec.^{2}; ang. acc. of DE = 12,800 rad./sec.^{2}.
11. 7·16 ft./sec.; −86·1 ft./sec.^{2}.
12. 1·20 rad./sec.; −8·34 rad./sec.^{2}.
14. Vel. of E = 18·8 in./sec.; acc. of E = −80 in./sec.^{2}.

Exercises V, pp. 83–88.

1. 1·61 ft./sec.^{2}; 24·15 ft./sec.; 5 sec.
2. 7·61 ft./sec.; 0·0328 sec.; (*a*) 18,000 lb.; (*b*) 20,500 lb.
4. $16\frac{2}{3}$ lb.; $33\frac{1}{3}$ lb.
5. 50 lb.; 75 lb.
6. 50·1
7. 63·5 m.p.h.
8. 9·1 ft.
9. 27·7.
10. 129·4 ft.; 0·706; 7·33 ft./sec.
11. 0·715.
12. 3·66 inches.
13. 11·6 inches.
14. 8·3 sec.
15. 38° 43′; 452 lb.
16. 3·64 inches.
17. 44,800; 273·3 sec.
18. 14·2.
19. 1·6 m.p.h.; 2·57 ft.-tons; 8·51.
20. (*a*) 15·2; (*b*) 51·8 lb.
21. 60 m.p.h.
22. $Wc \sin\theta + \mu W(a - c\cos\theta)$, taking $\theta = 0$ when weight is in lowest position.
23. $(Z - H_0)/W$.
24. $\{n_1 - wk(\lambda + \sin\alpha)/h\}k$; 12·5 m.p.h.
26. Per wheel: front +13·1 lb., back −13·1 lb.
27. Per wheel: outer +5·34 lb., inner −5·34 lb.

Exercises VII, pp. 114–116.

1. 0·327 sec.; 3·059 cycles/sec.
2. (i) $x = \frac{1}{8}\cos 4{\cdot}7\pi t$; (ii) $x = \frac{1}{8}\sin 4{\cdot}7\pi t$;
   (iii) $x = \frac{1}{16}\cos 4{\cdot}7\pi t - \frac{1}{16}\sqrt{3}\sin 4{\cdot}7\pi t = \frac{1}{8}\cos(4{\cdot}7\pi t + \frac{1}{3}\pi)$.
3. 1·63 ft.; 5·06 ft./sec.; 2·03 sec.
4. (i) 21·65 ft./sec.; (ii) 17·68 ft./sec.
5. 0·091 ft.; 0·955 ft./sec.
6. 89·0 ft./sec.; 22,370 ft./sec.^{2}; 7920 lb.
7. 0·452 sec.; 0·869 ft./sec.
8. 20·94 ft./sec.; 18·14 ft./sec.; 1316 ft./sec.^{2}; 133.
10. $4\pi^2$ ft./sec.^{2}.
11. 1·918 sec.; 0·013 sec.
12. 3·26 ft.; 86·3 sec.
13. 3·32 sec.; 6·64 sec.
14. 1·0174.

Exercises VIII, pp. 129–131.

1. $r = 3{\cdot}95$ inch; $\theta_1 = 13°\ 10'$; acceleration at A is 2924 ft./sec.^{2}; at B, 2847 ft./sec.^{2} and −625 ft./sec.^{2}; at D, −914 ft./sec.^{2}.
2. 0·822 inch.
3. (*a*) 1582 ft./sec.^{2}; (*b*) 0·0117, 0·0469, 0·1055, 0·1875, 0·2695, 0·3281, and 0·3633 inch; (*c*) see Fig. 203. Acceleration is shown here as positive

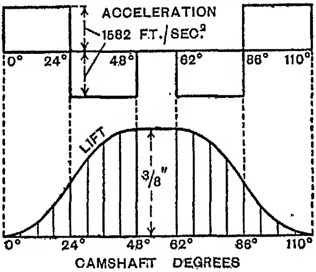

Fig. 203.

or negative according as its direction is that of valve opening or of valve closing.

4. (*a*) $\frac{1}{24}\cos\theta\left(\frac{d\theta}{dt}\right)^2$ ft./sec.^{2}; (*b*) 530 r.p.m.

5. (*a*) 594 r.p.m.; (*b*) 460 r.p.m.
6. 13° 4′; $\frac{299}{7200}$ or $\frac{1}{24}$ approx. 10. 23·4 lb.
11. 3·70 mm. 12. −990 ft./sec.^{2}; 23·1 lb.

Exercises IX, pp. 156–161.

1. (*a*) 1·51 ft./sec.^{2}; (*b*) 7·27 sec.; (*c*) 1·57 tons; 3·75 ton-ft.

2. 1·32 ton-ft.; 24·6. 3. (1) $\frac{\mu g b}{a+b-\mu h}$; (2) $\frac{\mu g a}{a+b+\mu h}$.

4. (*a*) 10·7 ft./sec.^{2}; (*b*) 0·192. 5. 13·7 sec. 6. 6·52 in.

7. (1) $\frac{3}{2}r$; (2) $\frac{17}{12}r$; (3) $\frac{3}{2}r$. 8. (1) $\frac{4\sqrt{2}}{3}a$; (2) $\frac{7\sqrt{2}}{6}a$; (3) $\frac{5}{4}a$.

9. $a=4·07$ in.; $b=9·68$ in.; $k=4·82$ in.
10. $a=6·16$ in.; $b=11·84$ in.; $k=6·63$ in.
11. $R_1=5·43$ lb.; $R_2=101·0$ lb.; 0·90 rad./sec.

13. $2\pi\left\{\frac{\frac{2}{5}r^2+l^2+x^2}{gx}\right\}^{\frac{1}{2}}$. 15. 2·51 ft.; 24·1 ft.-tons. 16. 14·4 ft./sec.

19. $\dot{\theta}=\left\{\frac{2mgb(\cos\theta-\cos\alpha)}{\frac{1}{2}M(k^2+a^2)+m(a^2-2ab\cos\theta+b^2)}\right\}^{\frac{1}{2}}$.

20. $N_1'=43·5$ r.p.m.; $N_2'=32·7$ r.p.m.; 0·28 in.-lb.

21. $N_1'=35·4$ r.p.m.; $N_2'=26·5$ r.p.m.; 31·7 in.-lb.

22. (i) 41·7 rad./sec.^{2}; 23·2 rad./sec.^{2}; (ii) 193 r.p.m.; 0·48 sec.

24. $\omega=\frac{\sqrt{70gh}}{7a-5h}$; $\frac{\sqrt{\omega^2(7a-5h)^2-70gh}}{7a}$.

27. $\frac{1}{2}ma^2\omega\sin^2\theta$.

Exercises X, pp. 201–205.

1. 0·815 sec. 2. 187·7; 159; 4·61 inches when using first spring, or 5·71 inches when using both springs.
3. 1·92 lb. 4. See Fig. 204.

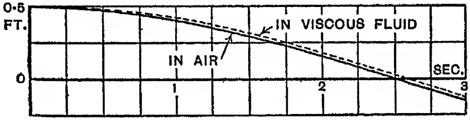

FIG. 204.

5. 1·78 sec.; 1·82 sec.; log. dec. $=0·682$; $R=6·75$ lb.
7. 6·29 in. 8. 0·0664 sec. 9. 26·3 oscillations/sec.

10. $T=2\pi\sqrt{\frac{32I}{C\pi}\int_0^l\frac{dx}{d^4}}=2\pi\sqrt{\frac{32Il}{3C\pi d_1^3d_2^3}(d_1^2+d_1d_2+d_2^2)}$.

11. 132 oscillations/sec. 12. 5·37 lb.-ft.; (i) 0·313m lb.; (ii) 0·199m lb.

13. (*a*) 1622 r.p.m.; (*b*) Total deflection = dynamic + static deflections = 0·000206 in. + 0·013393 in.

14. 0·627 sec.; (1) No damping, $y=3{\cdot}85\cos 10{\cdot}02t$, see Fig. 205;

(2) Damping, $y=e^{-7{\cdot}51t}(3{\cdot}85\cos\sqrt{44}t+4{\cdot}36\sin\sqrt{44}t)$, see Fig. 205;

$$\text{Amplitude}=\frac{\frac{1}{3}p^2\times 12}{\sqrt{(100{\cdot}4-p^2)^2+225{\cdot}6p^2}}\text{ in.,}\quad\text{see Fig. 206;}$$

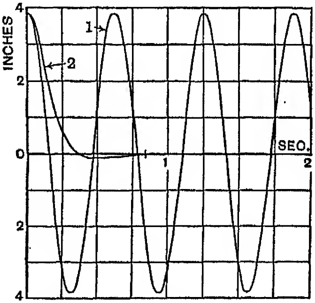

FIG. 205.

Angle of lag $\epsilon=\tan^{-1}\dfrac{15{\cdot}03p}{100{\cdot}4-p^2}$, see Fig. 207. Since p is in rad./sec., it is convenient to replace p by πN/30 when drawing the last two graphs, N being in r.p.m.

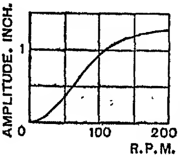

FIG. 206.

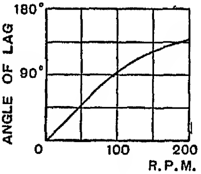

FIG. 207.

15. 16·0 oscillations/sec. 16. 481 r.p.m.

17. *First Method* (using deflections obtained when both loads are on shaft) 920 r.p.m.; *Second Method* 861 r.p.m. 18. 1950 r.p.m.

Exercises XI, pp. 237–242.

1. Disregarding signs, (*a*) $\dfrac{M_1 l}{EI}$; (*b*) $\dfrac{M_1 l^2}{2EI}$.

2. Disregarding signs, (*a*) $\dfrac{M_1 l}{2EI}$; (*b*) $\dfrac{M_1 l^2}{8EI}$.

3. Vertical, 0·665 inch; horizontal, 0·733 inch.

4. Slope $= \pm\frac{1}{EI}\left\{\frac{Wl^2}{16}+\frac{wl^3}{24}\right\}$; deflection $=\frac{1}{EI}\left\{\frac{Wl^3}{48}+\frac{5wl^4}{384}\right\}$.

5. (*a*) $\frac{5}{8}wl+3EI\frac{d}{l^3}$; (*b*) $\frac{3}{8}wl-3EI\frac{d}{l^3}$; (*c*) $\frac{1}{8}wl^2+3EI\frac{d}{l^2}$.

8. Refer to Fig. 184, p. 224.

Origin at A: $M=Px-W[x-\frac{1}{2}l]$, or $M=R(l-x)-W[\frac{1}{2}l-x]-M_1$.
Origin at B: $M=P(l-x)-W[\frac{1}{2}l-x]$, or $M=Rx-W[x-\frac{1}{2}l]-M_1$.

9. $R_1=5$ tons; $R_2=7$ tons; $x=15{\cdot}83$ feet; $y_{max}=1{\cdot}28$ inches.

10. $2{\cdot}12a$ approx.; $y_{max}=\frac{1{\cdot}19Wa^3}{EI}$ approx.

11. $AC=DB=(1-\frac{1}{3}\sqrt{3})l=0{\cdot}423l$ approx.

At centre, $y=\frac{(12\sqrt{3}-19)wl^4}{216EI}=\frac{wl^4}{121EI}$ approx.

At ends, $y=-\frac{(7-4\sqrt{3})wl^4}{54EI}=-\frac{wl^4}{752EI}$ approx.

15. Slope $=\frac{M_0l}{16EI}$; deflection $=\frac{Wl^3}{192EI}$.

20. See Fig. 208.

At (1), $M=9{\cdot}986$ ton-feet.
At (2), $M=6{\cdot}643$ ton-feet.
At (3), $M=8{\cdot}357$ ton-feet.

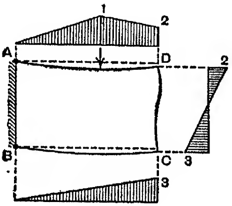

Fig. 208.

21. $y_{max}=\frac{12Wl^3}{Ebt^3}\left\{\frac{b}{2c}-\frac{b^2}{c^2}+\frac{b^3}{c^3}\log_e\left(\frac{b+c}{b}\right)\right\}$. This becomes indeterminate when $c=0$. Bring to a common denominator, differentiate numerator and denominator separately with respect to c, then put $c=0$ and the deflection becomes $y_{max}=\frac{4Wl^3}{Ebt^3}$.

22. 0·488 inch.

23. $\frac{112}{1312}$.

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